

Finite Dimensional KP τ -functions

I. Finite Grassmannians

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Abstract

We study τ -functions of the KP hierarchy in terms of abelian group actions on finite dimensional Grassmannians, viewed as subquotients of the Hilbert space Grassmannians of Sato, Segal and Wilson. A determinantal formula of Gekhtman and Kasman involving exponentials of finite dimensional matrices is shown to follow naturally from such reductions. All reduced flows of exponential type generated by matrices with arbitrary nondegenerate Jordan forms, are derived, both in the Grassmannian setting and within the fermionic operator formalism. A slightly more general determinantal formula involving resolvents of the matrices generating the flow, valid on the big cell of the Grassmannian, is also derived. An explicit expression is deduced for the Plücker coordinates appearing as coefficients in the Schur function expansion of the τ -function.

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1 Introduction

1.1 τ -functions and Hilbert space Grassmannians

In the approach to the KP integrable hierarchy developed by Sato [14, 15] and Segal and Wilson [16], all solutions are expressed in terms of the τ -function $\tau_W(\mathbf{t})$, which depends on the infinite set of KP flow parameters $\mathbf{t} = (t_1, t_2, \dots)$ and is parametrized by elements $W \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ of an infinite dimensional Grassmann manifold. The elements W are closed subspaces of a Hilbert space \mathcal{H} admitting a natural orthogonal splitting

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad (1.1)$$

into the direct sum of two semi-infinite subspaces \mathcal{H}_{\pm} , that are comparable with the subspace \mathcal{H}_+ . They are obtained by applying a bounded, invertible linear map $g \in GL(\mathcal{H})$ to the \mathcal{H}_+

$$W = g(\mathcal{H}_+). \quad (1.2)$$

In [16], \mathcal{H} is taken as the space $L^2(S^1)$ of square integrable functions $f(z)$ on the unit circle $|z| = 1$ in the complex plane, \mathcal{H}_+ and \mathcal{H}_- are the subspaces of functions with only positive or negative Fourier components, respectively. The sense in which the subspace $W \subset \mathcal{H}$ is comparable with \mathcal{H}_+ is that the orthogonal projection map $\pi_+^\perp : W \rightarrow \mathcal{H}_+$ to \mathcal{H}_+ is a Fredholm operator, while orthogonal projection $\pi_-^\perp : W \rightarrow \mathcal{H}_-$ to \mathcal{H}_- is compact.

The Grassmannian $\text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ is viewed as a universal phase space, with dynamics defined by the action of an infinite abelian group

$$\Gamma_+ = \{\gamma_+(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i \Lambda^i}\} \quad (1.3)$$

consisting of flows generated by shifts $\Lambda : z^i \rightarrow z^{i+1}$ of the orthonormal basis elements:

$$\begin{aligned} \Gamma_+ \times \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) &\rightarrow \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \\ (\gamma_+(\mathbf{t}), W) &\rightarrow W(\mathbf{t}) := \gamma_+(\mathbf{t})W. \end{aligned} \quad (1.4)$$

The flow parameters $\mathbf{t} = (t_1, t_2, \dots)$ are thus additive coordinates on the abelian group Γ_+ . The element $W \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ parametrizing the τ -function is the initial point $W(\mathbf{0})$ of the Γ_+ orbit $W(\mathbf{t})$ and $\tau_W(\mathbf{t})$ is defined as the determinant of the orthogonal projection of $W(\mathbf{t})$ to the subspace \mathcal{H}_+ :

$$\tau_W(\mathbf{t}) := \det(\pi_+^\perp : W(\mathbf{t}) \rightarrow \mathcal{H}_+). \quad (1.5)$$

relative to a suitably defined, admissible basis.

Conversely, knowing the τ -function is sufficient to determine W , since its Plücker coordinates $\pi_{\lambda,N}(W)$ are just the coefficients in the expansion of $\tau_W(\mathbf{t})$ in a basis of Schur functions

$$\tau_W(\mathbf{t}) = \sum_{\lambda} \pi_{\lambda,N}(W) S_{\lambda}(\mathbf{t}), \quad (1.6)$$

where (λ, N) denote a pair consisting of an integer partition λ and an integer $N \in \mathbf{Z}$. The latter is the Fredholm index of the orthogonal projection map $\pi_+^{\perp} : W \rightarrow \mathcal{H}_+$ to \mathcal{H}_+ , which determines the connected component of the Grassmannian $Gr_{\mathcal{H}_+}(\mathcal{H})$, and is referred to in [16] as the *virtual dimension* of W . The Plücker coordinates $\{\pi_{\lambda,N}(W)\}$ are not independent since they must satisfy the infinite set of quadratic Plücker relations. But, as shown by Sato ([14, 15]), these are equivalent to the Hirota bilinear differential relations for $\tau_W(\mathbf{t})$, which in turn are equivalent to the equations of the KP hierarchy. .

Remark 1.1 Gauge transformations. We recall that an invertible linear transformation of the form

$$W \mapsto \gamma_-(\mathbf{s})W, \quad \gamma_-(\mathbf{s}) := e^{\sum_{i=1}^{\infty} s_i z^{-i}} \quad (1.7)$$

has the effect of multiplying $\tau_W(\mathbf{t})$ by the linear exponential factor $e^{-\sum_{i=1}^{\infty} i s_i t_i}$.

$$\tau_{\gamma_-(\mathbf{s})W}(\mathbf{t}) = e^{-\sum_{i=1}^{\infty} i s_i t_i} \tau_W(\mathbf{t}). \quad (1.8)$$

Since the KP solutions are uniquely determined by the logarithmic derivatives of the corresponding Baker-Akhiezer function $\psi_W(z, \mathbf{t})$, given by the Sato formula

$$\psi_W(z, \mathbf{t}) = e^{\sum_{i=1}^{\infty} t_i z^i} \frac{\tau_W(\mathbf{t} - [z^{-1}])}{\tau_W(\mathbf{t})}, \quad [z^{-1}] := \left(\frac{1}{z}, \frac{1}{2z^2}, \dots, \frac{1}{iz^i}, \dots \right), \quad (1.9)$$

and the transformation (1.7) just multiplies $\psi_W(z, \mathbf{t})$ by the time independent factor $\gamma_-(\mathbf{s})$, this has no effect upon the solutions. These are therefore referred to as *gauge transformations*.

1.2 Gekhtman-Kasman finite determinantal formula

Gekhtman and Kasman [6, 7] found a very simple finite dimensional determinantal expression for a class of KP τ -functions in which the entries have exponential dependence on the flow parameters. These are determined by a triplet of matrices (A, B, C) in which A and C are $n \times N$ with $n < N$ and have maximal rank while B is a square $N \times N$ matrix. The finite determinantal formula

$$\tau_{(A,B,C)}^f(\mathbf{t}) = \det \left(A e^{\sum_{i=1}^{\infty} t_i B^i} C^T \right) \quad (1.10)$$

is easily shown to satisfy the Hirota bilinear relations of the KP hierarchy, provided the simple rank-1 condition

$$\text{rank}(AB(A^\perp)^T) \leq 1 \quad (1.11)$$

is satisfied, where A^\perp is any maximal rank $(N-n) \times N$ matrix whose rows are orthogonal to those of A . That is, they span the

$$k := N - n \quad (1.12)$$

dimensional orthogonal annihilator of the space spanned by the rows of A . For the τ -function not to vanish at the initial time $\mathbf{t} = \mathbf{0}$, we must also require that AC^T be nonsingular.

It will be useful to reformulate the rank-1 condition in a slightly different way. It is easy to see that (1.11) holds if and only if there exists an $n \times n$ matrix D and two vectors $\mathbf{f} \in \mathbf{C}^n$, $\mathbf{g} \in \mathbf{C}^N$ such that the equation

$$AB - D^T A = \mathbf{f}\mathbf{g}^T \quad (1.13)$$

is satisfied, i.e., that every row of AB can be expressed as a linear combination of the rows of A and the additional fixed vector \mathbf{g}^T .

The rank in (1.11) is 1 provided $\mathbf{f}\mathbf{g}^T$ is nonzero and \mathbf{g} does not belong to the row space of A . Otherwise $AB = \tilde{D}A$ for some matrix \tilde{D} and therefore

$$\tau_{(A,B,C)}^f(\mathbf{t}) = \det \left(A e^{\sum_{i=1}^{\infty} t_i B^i} C^T \right) = e^{\sum_{i=1}^{\infty} t_i \text{tr}(\tilde{D})^i} \det(AC^T), \quad (1.14)$$

which is gauge equivalent to a constant.

A slightly more general class of finite determinantal KP τ -functions of exponential type may be constructed as follows. For three positive integers l, n, N with $l \leq n$, $l \leq N$, we may again choose the matrices $D \in \text{Mat}^{n \times n}$, $B \in \text{Mat}^{N \times N}$ and a matrix $A \in \text{Mat}^{n \times N}$ satisfying the rank-1 condition (1.13) for some pair of vectors $\mathbf{f} \in \mathbf{C}^n$, $\mathbf{g} \in \mathbf{C}^N$. Then for any pair of rank- l matrices $F \in \text{Mat}^{l \times n}$, $C \in \text{Mat}^{l \times N}$, the following $l \times l$ determinant

$$\tau_{(A,B,C,D,F)}^f(\mathbf{t}) := \det(F e^{-\sum_{i=1}^{\infty} t_i (D^T)^i} A e^{\sum_{j=1}^{\infty} t_j B^j} C^T) \quad (1.15)$$

is a KP τ -function. The Gekhtman-Kasman formula (1.10), corresponds to the special case where $l = n$ and F is an invertible matrix, within the linear exponential factor gauge term $e^{-\sum_{i=1}^{\infty} t_i \text{tr}(D^T)^i}$. A simple direct proof that $\tau_{(A,B,C,D,F)}^f(\mathbf{t})$ satisfies the Hirota bilinear relations if the rank-1 condition eq. (1.13) is satisfied is given in the Appendix.

In the next subsection, some well-known examples expressible in the form (1.10) will be recalled. These include: all polynomial τ -functions, giving rise to rational solutions of the KP hierarchy; all nondegenerate multisoliton solutions, which generally are of exponential type; and all degenerations of the latter, in particular those that give rise to solutions that are rational in the $t_1 = x$ flow variable, with the locus of poles satisfying Calogero-Moser dynamics (cf. [12, 1, 17]). In the notation of Segal and Wilson [16], the rational solutions appearing in these examples belong to the sub-Grassmannian Gr_0 , while the multisoliton solutions and their degenerations belong to the sub-Grassmannian Gr_1 . Their place within the general setting is indicated in Section 1.4.

Section 2 gives a review of the fermionic approach to τ -functions. The general case of finite dimensional reductions leading to solutions of exponential or quasipolynomial type will be derived in detail in Sec. 3, both within the Grassmannian and the fermionic operator formalism. Sec. 4 gives a solution of the “inverse problem”; i.e., a reconstruction of the element $W(B, C, D)$ corresponding to any set (A, B, C, D) satisfying the rank-1 condition. These are viewed as a specialization of a more general class of finite dimensional τ -functions of exponential type, belonging to the big cell. The Plücker coordinates are explicitly determined for this general class, thereby determining the expansion of the τ -function in a basis of Schur functions.

1.3 Examples

Henceforth, \mathbf{I}_n denotes the $n \times n$ identity matrix and Λ_n the upper triangular shift matrix of size $n \times n$:

$$\Lambda_n := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \text{Mat}^{n \times n}. \quad (1.16)$$

Example 1.1 Rational solutions.

In formula (1.10), choose the following expressions for the matrices A and B

$$A = (\mathbf{I}_n \quad \mathbf{0}) \in \text{Mat}^{n \times N}, \quad B = \Lambda_N \in \text{Mat}^{N \times N}, \quad (1.17)$$

where $\mathbf{0}$ denotes the $n \times k$ matrix whose entries are all 0's. A basis for the orthogonal annihilator of the n -dimensional space spanned by the rows of A is given by the columns

of the $N \times k$ matrix

$$(A^\perp)^T := \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_k \end{pmatrix}. \quad (1.18)$$

We have

$$AB(A^\perp)^T = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \quad (1.19)$$

so the rank-1 condition (1.11) is satisfied. In the version (1.13) of the rank-1 condition, we have

$$D^T = \Lambda_n \in \text{Mat}^{n \times n}, \quad \mathbf{f}_a = \delta_{a,n}, \quad \mathbf{g}_b = \delta_{b,n+1}, \quad a = 1, \dots, n, \quad b = 1, \dots, N. \quad (1.20)$$

The matrix C can be any $n \times N$ matrix of maximal rank, which may be viewed as the homogeneous coordinates of an element $[C] \in \text{Gr}_n(\mathbf{C}^N)$ of the Grassmannian of n -dimensional subspaces of \mathbf{C}^N . For any partition λ whose Young diagram fits into that of the rectangular partition $(k)^n$ we let C_λ denote the $n \times n$ minor whose i th column is the $(\lambda_i - i + n + 1)$ th column of C . The corresponding Plücker coordinate $\pi_\lambda(C)$ of $[C] \in \text{Gr}_n(\mathbf{C}^N)$ is then

$$\pi_\lambda(C) = \det(C_\lambda). \quad (1.21)$$

It follows from the Cauchy-Binet identity that the expansion of the τ -function (1.10) in a basis of Schur functions $S_\lambda(\mathbf{t})$ is given by

$$\tau_{(A, \Lambda_N, C)}^f(\mathbf{t}) = \sum_{\lambda \subset (k)^n} \pi_\lambda(C) S_\lambda(\mathbf{t}). \quad (1.22)$$

This is the general form of KP τ -functions that have a polynomial dependence on all the KP flow parameters, which give rise to solutions of the hierarchy that are rational in all these variables.

Example 1.2 KP solitons.

Now choose B to be the diagonal matrix

$$B = B(\beta) := \text{diag}\{\beta_i\}_{i=1}^N \in \text{Mat}^{N \times N} \quad (1.23)$$

with distinct eigenvalues $\{\beta_i\}_{i=1\dots N}$, and A to be the truncated $n \times N$ Vandermonde matrix

$$A_V(\beta) := V_{n,N}(\beta) = \begin{pmatrix} \beta_1^{n-1} & \beta_2^{n-1} & \dots & \beta_N^{n-1} \\ \beta_1^{n-2} & \beta_2^{n-2} & \dots & \beta_N^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1 & \beta_2 & \dots & \beta_N \\ 1 & 1 & \dots & 1 \end{pmatrix} \quad (1.24)$$

Let

$$p(z) := \det(z\mathbf{I}_N - B) = \prod_{j=1}^N (z - \beta_j) \quad (1.25)$$

be the characteristic polynomial of B . It follows from the Cauchy residue theorem applied to

$$\frac{1}{2\pi i} \oint_{\infty} \frac{z^j dz}{p(z)} = 0, \quad \text{for } j < N - 1 \quad (1.26)$$

that the orthogonal complement of the subspace spanned by the rows of $A_V(\beta)$ is spanned by the rows of the $k \times N$ matrix

$$A_V(\beta)^\perp = \begin{pmatrix} \beta_1^{k-1} & \beta_2^{k-1} & \dots & \beta_N^{k-1} \\ \beta_1^{k-2} & \beta_2^{k-2} & \dots & \beta_N^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1 & \beta_2 & \dots & \beta_N \\ 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{p'(\beta_1)} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{p'(\beta_2)} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{p'(\beta_N)} \end{pmatrix}. \quad (1.27)$$

The rank-1 condition

$$A_V(\beta)B(A_V(\beta)^\perp)^T = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad (1.28)$$

follows from the Cauchy residue theorem applied to

$$\frac{1}{2\pi i} \oint_{\infty} \frac{z^{N-1} dz}{p(z)} = 1, \quad (1.29)$$

together with (1.26). The version (1.13) of the rank-1 condition is then satisfied, with

$$D = \Lambda_n, \quad \mathbf{f} := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{C}^n, \quad \mathbf{g} := \begin{pmatrix} \beta_1^n \\ \beta_2^n \\ \vdots \\ \beta_N^n \end{pmatrix} \in \mathbf{C}^N. \quad (1.30)$$

The resulting expression

$$\tau_{(A_V(\beta), B(\beta), C)}^f(\mathbf{t}) = \det \left(A_V(\beta) e^{\sum_{i=1}^{\infty} t_i B^i(\beta)} C^T \right) \quad (1.31)$$

$$= \sum_{\lambda \in (k)^n} \pi_{\lambda}(C) e^{T_{\lambda}(\beta, \mathbf{t})}, \quad (1.32)$$

where

$$T_{\lambda}(\beta, \mathbf{t}) := \sum_{i=1}^{\infty} t_i \sum_{j=1}^n \beta_{\ell_j}^i \quad (1.33)$$

$$\ell_j := \lambda_j - j + n + 1, \quad (1.34)$$

is the τ -function for the general rank n , N -soliton solution of the KP hierarchy. The second equality (1.32) follows from the Cauchy-Binet theorem applied to the product of the $n \times N$ and $N \times n$ matrices appearing in the determinant. The τ -function is real for real flow parameters $\mathbf{t} = (t_1, t_2, \dots)$ if the β_i 's and the matrix $C \in \text{Mat}^{n \times N}$ are real. It is nonvanishing, giving rise to nonsingular solutions, provided the β_i 's are strictly decreasing and C has only nonnegative Plücker coordinates, i.e., provided the space spanned by the rows of C belongs to the nonnegative Grassmannian $\text{Gr}_n^+(\mathbf{R}^N)$ [9, 10, 11].

Another variant of the determinantal form of the above solution may be obtained by choosing A as the Cauchy matrix:

$$A = A_C^0(\beta, \delta) := \begin{pmatrix} \frac{1}{\beta_1 - \delta_1} & \frac{1}{\beta_2 - \delta_1} & \cdots & \frac{1}{\beta_N - \delta_1} \\ \frac{1}{\beta_1 - \delta_2} & \frac{1}{\beta_2 - \delta_2} & \cdots & \frac{1}{\beta_N - \delta_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \delta_n} & \frac{1}{\beta_2 - \delta_n} & \cdots & \frac{1}{\beta_N - \delta_n} \end{pmatrix}. \quad (1.35)$$

In this case, the orthogonal annihilator is spanned by the rows of

$$(A_C^0(\beta, \delta))^{\perp} = \begin{pmatrix} \frac{r(\beta_1)}{(\beta_1 - \delta_{n+1})p'(\beta_1)} & \frac{r(\beta_2)}{(\beta_2 - \delta_{n+1})p'(\beta_2)} & \cdots & \frac{r(\beta_N)}{(\beta_N - \delta_{n+1})p'(\beta_N)} \\ \frac{r(\beta_1)}{(\beta_1 - \delta_{n+2})p'(\beta_1)} & \frac{r(\beta_2)}{(\beta_2 - \delta_{n+2})p'(\beta_2)} & \cdots & \frac{r(\beta_N)}{(\beta_N - \delta_{n+2})p'(\beta_N)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{r(\beta_1)}{(\beta_1 - \delta_N)p'(\beta_1)} & \frac{r(\beta_2)}{(\beta_2 - \delta_N)p'(\beta_2)} & \cdots & \frac{r(\beta_N)}{(\beta_N - \delta_N)p'(\beta_N)} \end{pmatrix}. \quad (1.36)$$

where $\{\delta_{n+1}, \dots, \delta_N\}$ is any further set of distinct constants, unequal to the previous δ_i 's or β_i 's and

$$r(z) := \prod_{i=1}^N (z - \delta_i). \quad (1.37)$$

This follows from Cauchy's theorem applied to

$$\frac{1}{2\pi i} \oint_{\infty} \frac{r(z)dz}{(z - \delta_j)(z - \delta_k)p(z)} = 0, \quad \text{for } 1 \leq j \leq n < k \leq N. \quad (1.38)$$

It also follows from Cauchy's theorem applied to

$$\frac{1}{2\pi i} \oint_{\infty} \frac{z r(z)dz}{(z - \delta_j)(z - \delta_k)p(z)} = 1 \quad (1.39)$$

that

$$A_C^0(\beta, \delta)B((A_C^0(\beta, \delta))^\perp)^T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \quad (1.40)$$

and hence has rank 1.

The version (1.13) of the rank-1 condition

$$A_C^0(\beta, \delta)B(\beta) - D^T A_C^0(\beta, \delta) = \mathbf{f}\mathbf{g}^T \quad (1.41)$$

is satisfied with

$$D = D(\delta) := \text{diag}\{\delta_i\}_{i=1}^n \in \text{Mat}^{n \times n} \quad (1.42)$$

and

$$\mathbf{f} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbf{C}^n, \quad \mathbf{g} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbf{C}^N. \quad (1.43)$$

In fact, the corresponding τ -function

$$\tau_{(A_C^0(\beta, \delta), B(\beta), C)}^f(\mathbf{t}) = \det \left(A_C^0(\beta, \delta) e^{\sum_{i=1}^{\infty} t_i (B(\beta))^i} C^T \right) \quad (1.44)$$

is of the same class as (1.31), and differs from it only by a slight modification of the choice of the matrix C . It therefore simply represents another parametrization of the multisoliton solutions, whatever the choice of the constants $\{\delta_i\}_{i=1, \dots, n}$.

To see this, define the $n \times n$ matrix

$$K_{ab}(\delta) = \frac{(\delta_b)^{n-a}}{r'(\delta_b)}, \quad 1 \leq a, b \leq n. \quad (1.45)$$

By evaluating the integral

$$(\beta_j)^a = \frac{1}{2\pi i} \oint_{|z|=1} \frac{z^{n-a}}{z - \beta_j} dz \quad (1.46)$$

using the Lagrange interpolation formula

$$z^{n-a} = \sum_{b=1}^n \frac{r(z)}{z - \delta_j} \frac{(\delta_b)^{n-a}}{r'(\delta_b)}, \quad (1.47)$$

we obtain the matrix product identity

$$A_V(\beta) = K(\delta) A_C^0(\beta, \delta) r(B(\beta)), \quad (1.48)$$

It follows from (1.48) that the τ -function of eq. (1.31) can be equivalently written as

$$\tau_{(A_V(\beta), B(\beta), C)}^f(\mathbf{t}) = \kappa(\delta) \tau_{(A_C^0(\beta, \delta), B, r(B(\beta))^T C)}^f(\mathbf{t}) \quad (1.49)$$

where

$$\kappa(\delta) := \det(K(\delta)), \quad (1.50)$$

since $r(B(\beta))$ commutes with $B(\beta)$. Thus $\tau_{(A_V(\beta), B(\beta), C)}^f(\mathbf{t})$ is just a multiple of $\tau_{(A_C^0(\beta, \delta), B, C)}^f(\mathbf{t})$ with C replaced by $r(B)^T C$. Since the choice of C is arbitrary, the two sets of τ -functions (1.31) and (1.44) coincide. To assure the reality and positivity condition however, it is necessary that the δ_i 's be real, and that all the entries of the diagonal matrix $r(B)$ be of the same sign. This will be satisfied if the δ_i 's are chosen to be less than all the β_i 's:

$$\delta_i < \beta_j, \quad i = 1, \dots, n, \quad j = 1, \dots, N. \quad (1.51)$$

Example 1.3 Generic case: degeneration of KP solitons.

More generally, both $B \in \text{Mat}^{N \times N}$ and $D \in \text{Mat}^{n \times n}$ may have any Jordan structure. Without loss of generality, we may choose them to be in standard upper triangular Jordan normal form, with distinct eigenvalues $\{\beta_j\}_{j=1, \dots, M}$ for B

$$B = \begin{pmatrix} J_{N_1}(\beta_1) & 0 & 0 & 0 \\ 0 & J_{N_2}(\beta_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_{N_M}(\beta_M) \end{pmatrix} \quad (1.52)$$

where

$$J_{N_j}(\beta_j) = \beta_j \mathbf{I}_{N_j} + \Lambda_{N_j}, \quad j = 1, \dots, M \quad (1.53)$$

denotes a Jordan block of dimension $\{N_j\}$, and eigenvalue β_j and D similarly is of the form

$$D = \begin{pmatrix} J_{n_1}(\delta_1) & 0 & 0 & 0 \\ 0 & J_{n_2}(\delta_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_{n_m}(\delta_m) \end{pmatrix} \quad (1.54)$$

with Jordan blocks of dimension $\{n_j\}$ and distinct eigenvalues $\{\delta_i\}_{i=1,\dots,m}$, also chosen to be distinct from the β_j 's. The equation

$$AB - D^T A = \mathbf{f} \mathbf{g}^T \quad (1.55)$$

then has a unique solution for any given pair of nonvanishing vectors $\mathbf{f} \in \mathbf{C}^n$, $\mathbf{g} \in \mathbf{C}^N$.

We may always multiply on the right by an element of the stabilizer $G_B \subset GL(N)$ of B under conjugation or on the left by an element of the stabilizer $G_D \subset GL(n)$ of D^T and obtain a new solution that gives on equivalent class of τ -functions. For such general B and D , the solution $A(B, D)$ to the rank-1 equation (1.55) for a suitable choice of \mathbf{f} and \mathbf{g} , is given in eq. (3.32) of in Sec. 3.2 and Proposition 3.4, with $r(z)$ and $p(z)$ replaced by the characteristic polynomials $r_D(z)$ and $r_B(z)$ of the matrices D and B respectively. Denoting by $A(B) := A(B, \Lambda_n)$ the special case when D is chosen as the shift matrix Λ_n , it follows, as in the above special case, that the τ -function

$$\tau_{(A(B,D), B, C)}^f(\mathbf{t}) = \det(A(B, D) e^{\sum_{i=1}^{\infty} t_i B^i} C^T) \quad (1.56)$$

determined by the triple $(A(B, D), B, C)$, as given by Theorem 3.5, coincides with $\tau_{(A(B), B, C)}^f(\mathbf{t})$ within a multiplicative constant. Therefore, the choice of D in the form of the rank-1 condition (1.55) does not affect the resulting class of solutions.

The next example is a special case of nondiagonal B , in which $N = 2n$, and B consists of n distinct 2×2 Jordan blocks, with a special choice of C , which gives rise to pole dynamics of the Calogero-Moser type.

Example 1.4 *Calogero-Moser pole dynamics.* ([12, 1, 17])

Choose B to be a $2n \times 2n$ matrix of the form

$$B = B_Z := \begin{pmatrix} Z & \mathbf{I}_n \\ \mathbf{0} & Z \end{pmatrix} \quad (1.57)$$

where Z is the diagonal $n \times n$ matrix

$$Z = \text{diag}\{\beta_i\}_{i=1}^n \quad (1.58)$$

with distinct eigenvalues $\{\beta_i\}_{i=1}^n$. For A , choose the modified truncated Vandermonde matrix

$$A_{V'}(\beta) := (V_{n,n}(\beta) \quad V'_{n,n}(\beta)) \quad (1.59)$$

where $V_{n,n}(\beta)$ is defined as in (1.24) and

$$V'_{n,n}(\beta) := \begin{pmatrix} (n-1)\beta_1^{n-2} & (n-1)\beta_2^{n-2} & \cdots & (n-1)\beta_n^{n-2} \\ (n-2)\beta_1^{n-3} & (n-2)\beta_2^{n-3} & \cdots & (n-2)\beta_n^{n-3} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (1.60)$$

and take C to be of the special form

$$C^T = C_\Xi^T := \begin{pmatrix} \mathbf{I}_n \\ \Xi \end{pmatrix}, \quad (1.61)$$

where

$$\Xi = \text{diag}\{\xi_i\}_{i=1}^n. \quad (1.62)$$

The rank-1 condition is easily verified by applying the Cauchy theorem to

$$\frac{1}{2\pi i} \oint_\infty \frac{z^j dz}{p^2(z)} = \delta_{j,2n} \quad (1.63)$$

where

$$\det(z \mathbf{I}_n - B_Z) = p^2(z) := \prod_{i=1}^n (z - \beta_i)^2. \quad (1.64)$$

The resulting τ -function is of the form

$$\tau_{(A_{V'}(\beta), B_Z, C_\Xi)}^f(\mathbf{t}) = e^{\sum_{i=1}^\infty t_i Z^i} \det(V_{n,n}(\beta)) \det\left(X_0 + \sum_{i=1}^\infty it_i Z^{i-1} \Xi\right) \quad (1.65)$$

where

$$X_0 = \mathbf{I}_n + V_{n,n}^{-1}(\beta) V'_{n,n}(\beta) \Xi, \quad (1.66)$$

which is gauge equivalent to the τ -function for rational solutions of the KP hierarchy in which the pole dynamics are determined by the Calogero-Moser n -particle system [12, 1, 17]. More general solutions in this class may be obtained by allowing the matrix Z to have general Jordan normal form, and Ξ to be an element of its centralizer.

1.4 Finite dimensional reductions of Grassmannians

The reduction to finite dimensional systems may be viewed as a two-step process: first the identification of a fixed subspace $W_2 \subset \mathcal{H}$, invariant under the flows, that contains W as a finite codimensional subspace. Second, a quotient by another fixed finite codimensional subspace $W_1 \subset W_2$, also invariant under the flows, that is contained in W . In the case where these subspaces are chosen as

$$W_1 = r(z)\mathcal{H}_+, \quad W_2 = \frac{r(z)}{p(z)}\mathcal{H}_+ \quad (1.67)$$

for a pair of polynomials $r(z), p(z)$ of degrees n and N , respectively, with the roots of both inside the unit circle, we obtain (within gauge equivalence) precisely the Grassmannian Gr_1 of [16]. The corresponding pair of matrices B and D are those whose eigenvalues coincide with the roots of $p(z)$ and $r(z)$, respectively, with Jordan blocks of dimension equal to the degree of these roots. This determines, up to conjugation, a unique pair of regular elements, $B \in \mathfrak{gl}(N)$, $D \in \mathfrak{gl}(n)$ whose characteristic polynomials are $p(z)$ and $r(z)$, respectively. Within gauge equivalence, there is no loss of generality in assuming that the roots of $r(z)$ and those of $p(z)$ are mutually distinct.

The finite dimensional reduction may be viewed as a subquotient. Projecting $W \rightarrow W/W_1$ gives an element of the finite dimensional Grassmannian $\text{Gr}_n(W_2/W_1)$ of n -dimensional subspaces of W_2/W_1 , and W_2/W_1 can be identified with \mathbf{C}^N through the choice of a suitable basis. The resulting flows can be expressed in terms of the Plücker coordinates of $W/W_1 \subset W_2/W_1$, and the corresponding KP τ -function becomes a finite determinant having linear exponential or quasi polynomial dependence on the flow variables. The generator of the reduced flow is a matrix B that may have any nondegenerate Jordan normal form, which is uniquely determined by the choice of basis for W_2/W_1 .

In particular, B could be nilpotent, consisting of a single Jordan block with zero eigenvalue; i.e., the $N \times N$ “shift” matrix Λ_N , whose characteristic polynomial is the monomial $p(z) = z^N$. This naturally gives rise to the polynomial τ -functions of Example 1.1 above. Alternatively, choosing $p(z)$ as the monic polynomial with distinct roots $\{\beta_i\}_{i=1,\dots,N}$ results in flows generated by the finite nondegenerate diagonal matrix with these eigenvalues, as in Example 1.2. The various other cases can be obtained by allowing multiple zeros in $p(z)$, which give rise to reduced flow generators B having all possible Jordan normal forms with distinct eigenvalues. Special cases of such degenerations of exponential or trigonometric multisoliton solutions may be used to embed certain finite dimensional integrable systems, such as the Calogero-Moser system of Example 1.4, as the dynamics of poles of rational solutions of the KP hierarchy.

The purpose of this paper is to provide a geometrical construction of such finite dimensional τ -functions through the process of reduction from the infinite case. We use subquotients to define families of embeddings of finite Grassmannians into infinite ones and deduce thereby the triplets (A, B, C) . The matrix A is determined by the choice of the fixed subspace W_1 , and the basis for a complement $W_1^c \subset W_2$ of W_1 in W_2 . The latter also determines the generating matrix B of the reduced flows. The element W , viewed as an extension of W_1 by a subspace of W_1^c determines the matrix C , and conversely. The projection $W \rightarrow W/W_1 \subset W_2/W_1$ allows us to identify C as the homogeneous coordinates of an initial point in the finite Grassmannian $\text{Gr}_n(W_2/W_1)$ which is identified, through the choice of basis for W_1^c , as a subspace of \mathbf{C}^N .

Remark 1.2 There are other instances of τ -functions expressible as finite dimensional determinants, in which there is no known interpretation in terms of finite dimensional Grassmannians. For instance, the partition function in random matrix models, in which the underlying conjugation invariant measure is subject to linear exponential deformations, is known to be a KP τ -function that admits a finite dimensional determinantal representation in terms of the Hankel matrix formed from the moments. However, this does not seem to fit within the finite dimensional reduction framework discussed here, since the dependence upon the flow parameters is not exponential or quasipolynomial.

Other cases, such as the solutions of the KP-hierarchy expressible in terms of Riemann θ -functions on the Jacobi varieties of an algebraic curve, also involve a reduction to a finite number of degrees of freedom [13, 5]. However, the resulting τ -function is not known to be expressible as a finite dimensional determinant.

2 KP τ -functions

2.1 Grassmannians and fermionic Fock space

Following Segal and Wilson [16], the model for the Hilbert space \mathcal{H} we use is the space $L^2(S^1)$ of square integrable functions on the unit circle $S^1 = \{z \in \mathbf{C}, |z| = 1\}$ in the complex z -plane. This splits into the direct sum

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \quad (2.1)$$

of subspaces \mathcal{H}_+ and \mathcal{H}_- , consisting, respectively, of functions admitting a holomorphic continuation to the interior and exterior of the unit circle, with the latter vanishing at ∞ . These may be viewed as completions of the span of the positive and negative monomials in z

$$\mathcal{H}_+ = \overline{\text{span}\{z^i\}_{i \in \mathbf{N}}}, \quad \mathcal{H}_- = \overline{\text{span}\{z^{-i}\}_{i \in \mathbf{N}^+}}. \quad (2.2)$$

For consistency with other conventions, it is convenient to label the monomial basis as

$$e_i := z^{-i-1}, \quad i \in \mathbf{Z}. \quad (2.3)$$

Then \mathcal{H}_+ and \mathcal{H}_- are mutually orthogonal with respect to the complex inner product (\cdot, \cdot) in which these are orthonormal

$$(e_i, e_j) = \delta_{ij}. \quad (2.4)$$

The elements of the Grassmannian $\text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ are subspaces $W \subset \mathcal{H}$ that are comparable with \mathcal{H}_+ , in the sense that orthogonal projection to \mathcal{H}_+

$$\pi_+^\perp : W \rightarrow \mathcal{H}_+ \quad (2.5)$$

along \mathcal{H}_- is a Fredholm operator, while projection to \mathcal{H}_- along \mathcal{H}_+

$$\pi_-^\perp : W \rightarrow \mathcal{H}_- \quad (2.6)$$

is compact. The Fredholm index N of the projection map $\pi_+^\perp : W \rightarrow \mathcal{H}_+$ is called the “virtual dimension” of W . The subspace

$$\mathcal{H}_+^N := z^{-N} \mathcal{H}_+ \subset \mathcal{H}, \quad (2.7)$$

in particular, has virtual dimension N . The connected components of $\text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ consist of those $W \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ with virtual dimension $N \in \mathbf{Z}$. These may be viewed as the orbit $\text{Gr}_{\mathcal{H}_+^N}(\mathcal{H})$ of \mathcal{H}_+^N under the identity component $GL_0(\mathcal{H})$ of the restricted infinite dimensional Lie group $GL_{res}(\mathcal{H})$ of invertible linear transformations of \mathcal{H} having a well-defined determinant and preserving the properties defining the elements of $\text{Gr}_{\mathcal{H}_+}(\mathcal{H})$. (See [16] for more detailed definitions.)

2.2 Fermionic Fock space and the Plücker embedding

The fermionic Fock space \mathcal{F} is defined as the semi-infinite exterior space

$$\mathcal{F} = \bigwedge \mathcal{H} \quad (2.8)$$

spanned by an orthonormal basis $\{|\lambda; N\rangle\}$ consisting of semi-infinite wedge products

$$|\lambda; N\rangle := e_{l_1} \wedge e_{l_2} \wedge \cdots, \quad (2.9)$$

where $\{l_1, l_2, \dots\}$ is a strictly decreasing sequence of integers $l_1 > l_2 > \dots$, eventually stabilizing on a consecutive sequence of decreasing integers. The partition

$$\lambda := \{\lambda_1 \geq \lambda_2 \geq \lambda_{\ell(\lambda)} > 0\} \quad (2.10)$$

of length $\ell(\lambda)$, is related to the sequence by

$$l_i := \lambda_i - i + N \quad (2.11)$$

(with the convention that $\lambda_i = 0$ for $i > \ell(\lambda)$). The integer N is the largest one below the $\frac{1}{2}$ -integer point $\nu \in \mathbf{Z} + \frac{1}{2}$ on the real line such that, if all integer sites $\{l_i\}$ are viewed as “occupied” and all others as unoccupied, there are as many unoccupied sites to the left of ν as there are occupied sites to the right. The fermionic Fock space \mathcal{F} thus admits a decomposition

$$\mathcal{F} = \bigoplus_{N \in \mathbf{Z}} \mathcal{F}_N \quad (2.12)$$

as an orthogonal direct sum of the subspaces \mathcal{F}_N spanned by basis elements of charge N . The basis element

$$|0; N\rangle = e_{N-1} \wedge e_{N-2} \wedge \dots := |N\rangle \in \mathcal{F}_N \quad (2.13)$$

is referred to as the charge N vacuum state, and denoted simply as $|N\rangle$. (The reason for the seemingly reversed sign convention in (2.3) is that, under the Plücker map, the element $\mathcal{H}_+ \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ should correspond to the vacuum element $|0\rangle$, which is the “Dirac sea”, in which all the *negative* integer lattice sites are occupied.)

As in finite dimensions, the Grassmannian $\text{Gr}_{\mathcal{H}_+}(\mathcal{H})$, and each of its connected components $\text{Gr}_{\mathcal{H}_+^N}(\mathcal{H})$, may be viewed as infinite dimensional analogs of algebraic varieties, since they can be embedded into the projectivization $\mathbf{P}(\mathcal{F})$ by the Plücker map

$$\begin{aligned} \mathfrak{Pl} : \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) &\rightarrow \mathbf{P}(\mathcal{F}) \\ \mathfrak{Pl} : \text{span}\{w_1, w_2, \dots\} &\mapsto [w_1 \wedge w_2 \wedge \dots], \end{aligned} \quad (2.14)$$

where $[\dots]$ denotes the projective class, and the image $\mathfrak{Pl}(\text{Gr}_{\mathcal{H}_+}(\mathcal{H})) \subset \mathbf{P}(\mathcal{F})$ is the intersection of an infinite number of quadrics, defined by the Plücker relations. It follows from (2.14) that the image $\mathfrak{Pl}(\text{Gr}_{\mathcal{H}_+}(\mathcal{H})) \subset \mathbf{P}(\mathcal{F})$ consists of all decomposable elements in $\mathbf{P}(\mathcal{F})$, while the image of the virtual dimension N component $\text{Gr}_{\mathcal{H}_+^N}(\mathcal{H})$ is in $\mathbf{P}(\mathcal{F}_N)$. In particular, the image $\mathfrak{Pl}(\mathcal{H}_+)$ of the element $\mathcal{H}_+ \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ is (the projectivization of) the vacuum element $|0\rangle$.

$$\mathfrak{Pl} : \mathcal{H}_+ \mapsto [|0\rangle] \quad (2.15)$$

From the definition of the Plücker map and the scalar product on \mathcal{F} , it follows that the Plücker coordinates

$$\pi_{\lambda,N}(W) := \langle \lambda; N | \mathfrak{Pl}(W) \rangle \quad (2.16)$$

are determinants of the semi-infinite matrices that appear as maximal minors of the matrix of homogeneous coordinates of $W \in Gr_{\mathcal{H}_+}(\mathcal{H})$ relative to the given orthonormal basis. In what follows, it will be sufficient to consider only elements $W \in Gr_{\mathcal{H}_+}(\mathcal{H})$ that have virtual dimension 0 and hence, unless otherwise needed, the index N labelling the Plücker coordinate will be understood to be 0, the basis states $|\lambda; 0\rangle$ denoted simply as $|\lambda\rangle$ and the Plücker coordinates as

$$\pi_{\lambda}(W) := \langle \lambda | \mathfrak{Pl}(W) \rangle \quad (2.17)$$

The determinantal formula (1.5) defining the τ -function may be interpreted as the Plücker coordinate of the element $W(\mathbf{t})$ corresponding to the trivial partition,

$$\tau_W(\mathbf{t}) = \pi_0(W(\mathbf{t})). \quad (2.18)$$

Relative to the basis $\{e_i\}_{i \in \mathbf{Z}}$, we have the standard cellular decomposition, in which the “big cell” consists of all elements $W \in Gr_{\mathcal{H}_+}(\mathcal{H})$ that can be represented as the graph of a linear map $\mathcal{A} : \mathcal{H}_+ \rightarrow \mathcal{H}_-$. A basis for such an element may be chosen to consist of elements of the form

$$w_i := e_{-i-1} + \sum_{j=0}^{\infty} \mathcal{A}_{ij} e_j, \quad i \in \mathbf{N} \quad (2.19)$$

where the elements $\{\mathcal{A}_{ij}\}_{i,j \in \mathbf{N}}$ of the semi-infinite matrix \mathcal{A} are standard affine coordinates on the big cell. It follows from the definition of the Plücker coordinates that these coincide, within a sign, with the Plücker coordinates corresponding to hook partitions $(i+1, (1)^j)$ which, in Frobenius notation are denoted $(i|j)$

$$\mathcal{A}_{ij} = (-1)^b \pi_{(i|j)}(W) \quad (2.20)$$

More generally, denoting a partition λ in Frobenius notation as $(a_1, \dots, a_r | b_1, \dots, b_r)$, where a_i is the number of boxes to the right of the (i, i) diagonal element of the Young diagram and b_i the number of elements beneath it, the Plücker coordinate $\pi_{(a_1, a_2, \dots, a_k | b_1, b_2, \dots, b_k)}$ may be expressed, on the coordinate neighborhood of the big cell, in terms of those for the hook partitions through a generalized Giambelli formula [8]:

$$\pi_{(a_1, a_2, \dots, a_k | b_1, b_2, \dots, b_k)} = (-1)^{\sum_{i=1}^k b_i} \det(\mathcal{A}_{a_i, b_j}) \quad (2.21)$$

The image $\mathfrak{Pl}(W(\mathbf{t}))$ of the Γ_+ orbit $W(\mathbf{t})$ may be simply expressed in terms of fermionic creation and annihilation operators $\{\psi_i, \psi_i^\dagger\}_{i \in \mathbf{Z}}$ defined, respectively, as exterior products with the basis elements $\{e_i\}$, and interior products with the dual basis element $\{\tilde{e}^i\}$.

$$\psi_i := e_i \wedge, \quad \psi_i^\dagger := \tilde{e}^i \lrcorner, \quad i \in \mathbf{Z}. \quad (2.22)$$

These satisfy the usual anticommutation relations

$$[\psi_i, \psi_j]_+ = [\psi_i^\dagger, \psi_j^\dagger]_+ = 0, \quad [\psi_i, \psi_j^\dagger]_+ = \delta_{ij}. \quad (2.23)$$

and span the subspace of linear elements of the Clifford algebra associated to the group of orthogonal transformations $O(\mathcal{H} + \mathcal{H}^*, Q)$ preserving the natural quadratic form

$$Q(X, \nu) = 2\nu(X), \quad X \in \mathcal{H}, \quad \nu \in \mathcal{H}^* \quad (2.24)$$

on the sum $\mathcal{H} + \mathcal{H}^*$ of the underlying Hilbert space and its analytic dual. We also have the fermionic field operator $\psi(z)$, and its dual $\psi^\dagger(z)$,

$$\psi(z) := \sum_{i \in \mathbf{Z}} \psi_i z^i, \quad \psi^\dagger(z) := \sum_{i \in \mathbf{Z}} \psi_i^\dagger z^{-i-1}, \quad (2.25)$$

which may be viewed as generating functions for the ψ_i 's and ψ_i^\dagger 's.

The subgroup $GL(\mathcal{H}) \subset O(\mathcal{H} + \mathcal{H}^*, Q)$ of general linear transformations $GL(\mathcal{H})$, and its abelian subgroup $\Gamma_+ \subset GL(\mathcal{H})$, generating the commuting KP flows act naturally on the exterior space through the fermionic representation.

$$g := e^\xi \mapsto \hat{g} := e^{\sum_{i,j \in \mathbf{Z}} \xi_{ij} \psi_i \psi_j^\dagger} \quad (2.26)$$

where ξ_{ij} are the matrix components of the Lie algebra element $\mathcal{A} \in \text{End}(\mathcal{H})$ in the $\{e_i\}$ basis. In this notation the fermionic representation of the elements $\gamma_+(\mathbf{t}) \in \Gamma_+$ defining the KP flows is

$$\hat{\gamma}_+(\mathbf{t}) = e^{\sum_{i \in \mathbf{Z}} t_i J_i} \quad (2.27)$$

where

$$J_i := \sum_{k \in \mathbf{Z}} \psi_k \psi_{k+i}^\dagger, \quad i \in \mathbf{N}^+. \quad (2.28)$$

are the generators of the “shift” flows in the fermionic representation (which are Fourier components of the *current* operator).

It follows that the KP τ -function $\tau_W(\mathbf{t})$ may equivalently be represented as the vacuum state expectation value of a product of such group elements

$$\tau_W(\mathbf{t}, N) = \langle N | \hat{\gamma}_+(\mathbf{t}) \hat{g} | N \rangle, \quad (2.29)$$

where $g \in GL_0(\mathcal{H})$ is any element that takes \mathcal{H}_+^N into W

$$W = g(\mathcal{H}_+), \quad (2.30)$$

and N is the Fredholm index of the projection map $\pi_+^\perp : W \rightarrow \mathcal{H}_+$. Eq. (2.29) may be understood as defining the $|N\rangle$ component of $\mathfrak{Pl}(W(\mathbf{t}))$ (which is nonzero only if W has virtual dimension N); i.e., the Plücker coordinate $\pi_{0,N}(W(\mathbf{t}))$ of the moving point $W(\mathbf{t})$ under the KP flows, and is thus given, up to projectivization, by the semi-infinite determinant (1.5).

More generally, \hat{g} need not be a $GL(\mathcal{H})$ group element; it may be any element of the Clifford algebra satisfying the bilinear relation

$$\left[\sum_{i \in \mathbf{Z}} \psi_i \otimes \psi_i^\dagger, \hat{g} \otimes \hat{g} \right] = 0 \quad (2.31)$$

acting upon $\mathcal{F} \otimes \mathcal{F}$. Eq. (2.31) is equivalent to the Plücker relations and guarantees that $\hat{g}|0\rangle$ is a decomposable element, as in (2.14). In particular, (2.31) is satisfied by any product of pure creation or annihilation operators of the form

$$w_a := \sum_{i \in \mathbf{Z}} w_{ai} \psi_i, \quad v_a^\dagger = \sum_{i \in \mathbf{Z}} v_{ai} \psi_i^\dagger. \quad (2.32)$$

More generally, we have the following useful result.

Lemma 2.1 *For any number of creation and annihilation operators $\{w_a, v_a^\dagger\}_{a=1 \dots n}$, if an element \hat{g} satisfies the bilinear identity (2.31), so does the product*

$$\left(\prod_{a=1}^n w_a \right) \left(\prod_{b=1}^n v_b^\dagger \right) \hat{g}, \quad (2.33)$$

and hence

$$\tau_{(\mathbf{w}, \mathbf{v}, g)} := \langle N | \hat{\gamma}_+(\mathbf{t}) \left(\prod_{a=1}^n w_a \right) \left(\prod_{b=1}^n v_b^\dagger \right) \hat{g} | N \rangle \quad (2.34)$$

is a KP τ -function.

Proof. It follows immediately from the definitions that, if any two operators satisfy the bilinear identity, so does their product. Therefore, it is sufficient to prove it holds for any creation operator w_a or any annihilation operator v_a^\dagger . Now, let $\mu, \nu \in \mathcal{F}$ be a pair of elements and apply the product $\left(\sum_{i \in \mathbf{Z}} \psi_i \otimes \psi_i^\dagger\right)(w \otimes w)$ to the decomposable element $\mu \otimes \nu \in \mathcal{F} \otimes \mathcal{F}$.

$$\begin{aligned}
\sum_{i \in \mathbf{Z}} (\psi_i \otimes \psi_i^\dagger)(w_a \otimes w_a) \mu \otimes \nu &= \sum_{i \in \mathbf{Z}} \psi_i w_a \mu \otimes \psi_i^\dagger w_a \nu \\
&= \sum_{i \in \mathbf{Z}} w_a \psi_i \mu \otimes w_a \psi_i^\dagger \nu - \sum_{i \in \mathbf{Z}} \psi_i w_a \mu \otimes w_{ai} \nu \\
&= \sum_{i \in \mathbf{Z}} w_a \psi_i \mu \otimes w_a \psi_i^\dagger \nu - (w_a)^2 \mu \otimes \nu \\
&= \sum_{i \in \mathbf{Z}} w_a \psi_i \mu \otimes w_a \psi_i^\dagger \nu,
\end{aligned} \tag{2.35}$$

where the anticommutation relations (2.23) have been used, and the fact that $w_a^2 = 0$. Therefore the bilinear relation (2.31)

$$\left[\sum_{i \in \mathbf{Z}} \psi_i \otimes \psi_i^\dagger, w_a \otimes w_a \right] = 0 \tag{2.36}$$

is satisfied by w_a . A similar calculation shows it holds for v_b^\dagger , and hence for all products of the form (2.33). **Q.E.D.**

3 Reducing infinite to finite Grassmannians

3.1 Grassmannian subquotients $W_1 \subset W \subset W_2 \rightarrow W_2/W_1$

We now detail the subquotient reduction described above. The first step consists of choosing a pair of subspaces

$$W_1 \subset W_2 \subset \mathcal{H}, \quad W_1, W_2 \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \tag{3.1}$$

of virtual dimension $(-n)$ and k respectively, with $n + k = N$, both invariant under the action of the abelian group Γ_+ of KP flows

$$\gamma_+(\mathbf{t})W_1 \subset W_1, \quad \gamma_+(\mathbf{t})W_2 \subset W_2, \quad \forall \gamma_+(\mathbf{t}) \in \Gamma_+ \tag{3.2}$$

so that

$$\dim W_2/W_1 = n + k = N. \quad (3.3)$$

The element $W \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ is chosen to belong to the sub-Grassmannian of virtual dimension 0 subspaces that fit between them

$$W_1 \subset W \subset W_2. \quad (3.4)$$

Thus

$$\dim(W/W_1) = n, \quad \dim(W_2/W) = k. \quad (3.5)$$

We now make an identification of the quotient W_2/W_1 with \mathbf{C}^N by choosing a subspace $W_1^c \subset W_2$ that is complementary to $W_1 \subset W_2$, choosing a basis $\{b_1, b_2, \dots, b_N\}$ for it, and identifying this with the standard basis $\{f_1, f_2, \dots, f_N\}$ for \mathbf{C}^N

$$(f_i)_j = \delta_{ij}, \quad 1 \leq i, j \leq N. \quad (3.6)$$

Through the quotient map

$$W_2 \rightarrow W_2/W_1 \equiv \mathbf{C}^N, \quad (3.7)$$

any element $W \in \text{Gr}_{\mathcal{H}_+^0}(\mathcal{H})$ containing W_1 and contained in W_2 can be associated with a unique maximal rank $n \times N$ matrix C such that

$$W = W_1 \oplus \text{span} \left\{ \sum_{i=1}^N C_{ai} b_i \right\}, \quad a = 1, \dots, n. \quad (3.8)$$

The projection

$$W \rightarrow W/W_1 \subset W_2/W_1 \equiv \mathbf{C}^N \quad (3.9)$$

thus defines an element $[C]$ of the Grassmannian $\text{Gr}_n(\mathbf{C}^N)$, spanned by the rows of C . We denote this finite dimensional subquotient

$$\text{Gr}_n(W_2/W_1) \sim \text{Gr}_n(\mathbf{C}^N) \quad (3.10)$$

Since the flow group Γ_+ leaves both W_1 and W_2 invariant, this induces an action of Γ_+ on $\text{Gr}_n(W_2/W_1)$ such that, through the identification $W_2/W_1 \sim \mathbf{C}^N$, the shift map $\Lambda : \mathcal{H} \rightarrow \mathcal{H}$ may be represented by an $N \times N$ matrix

$$B : \mathbf{C}^N \rightarrow \mathbf{C}^N \quad (3.11)$$

whose form depends on the choice of this basis, but whose Jordan canonical form depends only on the choice of pairs (W_1, W_2) . The Γ_+ action induced on $\text{Gr}_n(\mathbf{C}^N)$ will then be represented by

$$\gamma_+(\mathbf{t}) : C^T \mapsto e^{\sum_{i=1}^{\infty} t_i B^i} C^T := C^T(\mathbf{t}). \quad (3.12)$$

To determine the associated τ -function $\tau_W(\mathbf{t})$, we must evaluate the Plücker coordinate $\pi_0(W(\mathbf{t}))$, which is the determinant of the projection operator

$$W(\mathbf{t}) \rightarrow \mathcal{H}_+. \quad (3.13)$$

If W_1 is chosen to be a subspace of \mathcal{H}_+ , which in turn is contained in W_2

$$W_1 \subset \mathcal{H}_+ \subset W_2, \quad (3.14)$$

we may view the Grassmannian $\text{Gr}_n(W_2/W_1)$ as an orbit of the element $\mathcal{H}_+/W_1 \in \text{Gr}_n(W_2/W_1)$. The Plücker coordinate $\pi_0(C(\mathbf{t}))$ relative to the given basis then coincides with $\pi_0(W(\mathbf{t}))$, and we may proceed in the same way as on the infinite Grassmannian $\text{Gr}_{\mathcal{H}_+^0}(\mathcal{H})$. If, however, the inclusion condition

$$\mathcal{H}_+ \subset W_2 \quad (3.15)$$

is not satisfied, a further transformation is needed to identify the reduced Grassmannian $\text{Gr}_n(W_2/W_1)$ as the orbit of some standard element under $GL(W_2/W_1)$. This transformation determines the matrix A in the Gekhtman-Kasman formula ([6, 7]).

We now consider the case when W_1, W_2 are defined to be

$$W_1 := r(z)\mathcal{H}_+, \quad W_2 := r(z)/p(z)\mathcal{H}_+, \quad (3.16)$$

where $r(z)$ and $p(z)$ are monic polynomials of degrees n and N respectively, with roots and multiplicities $\{\delta_i, n_i\}_{i=1, \dots, m}$, $\{\beta_j, N_j\}_{j=1, \dots, M}$

$$\begin{aligned} r(z) &:= \prod_{i=1}^m (z - \delta_i)^{n_i}, & p(z) &:= \prod_{j=1}^M (z - \beta_j)^{N_j}, \\ \sum_{i=1}^m n_i &= n, & \sum_{j=1}^M N_j &= N, \end{aligned} \quad (3.17)$$

with the roots $\{\delta_i\}$ and $\{\beta_j\}$ of both $r(z)$ and $p(z)$ chosen to lie within the unit circle. (If they do not, we may just redefine the circle S^1 in $\mathcal{H} = L^2(S^1)$ as having a sufficiently large

radius that all roots of are in the interior.) W_1 is thus the subspace of \mathcal{H}_+ consisting of elements that vanish at the roots $\{\delta_i\}_{i=1,\dots,d}$ of $r(z)$ to the same order as their multiplicities $\{n_i\}_{i=1,\dots,d}$ in $r(z)$, while W_2 is the direct sum of W_1 with the span of the rational basis elements

$$b_{(j\nu)}(z) := \frac{r(z)}{(z - \beta_j)^\nu}, \quad 1 \leq j \leq M, \quad 1 \leq \nu \leq N_j. \quad (3.18)$$

Note that if we use this basis to identify the quotient space W_2/W_1 with \mathbf{C}^N , the matrix B representing multiplication by z is precisely the Jordan normal form matrix defined in (1.52).

To complete the explicit matrix representation of the flows and Plücker coordinates, we must identify the standard basis for \mathbf{C}^n with a suitably chosen basis for \mathcal{H}_+/W_1 . We could of course choose this as the monomials of degree less than n , modulo W_1 . But a more convenient choice consists of

$$d_{(i\mu)}(z) := \frac{z^{\mu-1}}{(1 - z\delta_i)^\mu}, \quad 1 \leq i \leq m, \quad 1 \leq \mu \leq n_i. \quad (3.19)$$

These are linearly independent elements of \mathcal{H}_+ since the roots δ_i of $r(z)$ are distinct and lie within the unit circle. Moreover the $d_{(i\mu)}(z)$'s are orthogonal to $W_1 = r(z)\mathcal{H}_+$ with respect to the complex inner product (\cdot, \cdot) on \mathcal{H} in which the monomials are orthonormal, which may be expressed by the contour integral:

$$(f, g) := \frac{1}{2\pi i} \oint_{|z|=1} f(z^{-1})g(z) \frac{dz}{z}, \quad (3.20)$$

since, by the Cauchy theorem,

$$(d_{(i\mu)}(z), r(z)z^a) = 0, \quad \forall a \in \mathbf{N}. \quad (3.21)$$

They therefore form a basis for the orthogonal complement $W_1^\perp \subset \mathcal{H}_+$. The pairs (i, μ) may more concisely be labelled

$$I := (i, \mu), \quad 1 \leq i \leq m, \quad 1 \leq \mu \leq n_i \quad (3.22)$$

ordered consecutively as

$$(1, 1), \dots (1, n_1), \dots (i, 1), \dots (i, n_i), \dots (m, 1), \dots (m, n_m). \quad (3.23)$$

We assume henceforth that the roots $\{\delta_i\}_{i=1,\dots,m}$ and $\{\beta_j\}_{j=1,\dots,M}$ of the polynomials $r(z)$ and $p(z)$ are *all* distinct and interpret these as characteristic polynomials of the pair

of matrices $B \in \text{Mat}^{N \times N}$, $D \in \text{Mat}^{n \times n}$ defined in (1.52) and (1.54),

$$p(z) = r_B(z) := \det(z \mathbf{I}_N - B) = \prod_{j=1}^M (z - \beta_j)^{N_j}, \quad (3.24)$$

$$r(z) = r_D(z) := \det(z \mathbf{I}_n - D) = \prod_{i=1}^m (z - \delta_i)^{n_i}. \quad (3.25)$$

Denoting the pairs of indices $\{(j, \nu)\}$ by

$$J := (j, \nu), \quad 1 \leq j \leq M, \quad 1 \leq \nu \leq N_j, \quad (3.26)$$

ordered again consecutively as

$$(1, 1), \dots, (1, N_1), \dots, (j, 1), \dots, (j, N_j), \dots, (M, 1), \dots, (M, N_M). \quad (3.27)$$

The basis elements for W_2/W_1 , labelled accordingly, will be denoted b_J , and the elements of the $n \times N$ matrix $C \in \text{Mat}^{n \times N}$ as $C_{a,J}$ with $a = 1, \dots, n$ or, when a refinement is needed, as $C_{I,J}$, with I defined as in (3.22). The sub-Grassmannian $\text{Gr}_n(W_2/W_1)$ then consists of all W 's of the form

$$W(B, C, D) := W_1 \oplus \text{span} \left(\sum_J C_{I,J} b_J \right)_{1 \leq a \leq n}. \quad (3.28)$$

Remark 3.1 Gauge equivalence. Gauge transformations (1.7) that preserve the class of subspaces W_1, W_2 of type (3.16) consist of multiplication by rational functions that take value 1 at $z = \infty$; i.e., the ratio of two monic polynomials $q(z), \tilde{q}(z)$ of the same degree

$$\gamma_-(\mathbf{s}) = \frac{q(z)}{\tilde{q}(z)}, \quad \deg(q) = \deg(\tilde{q}). \quad (3.29)$$

We can therefore use gauge transformations to replace $r(z)$ by any rational function whose singular part at $z = \infty$ is a polynomial of degree n .

Remark 3.2 The union of all sub-Grassmannians $\text{Gr}_n(W_2/W_1)$ over all choices of (n, N) , $N > n$, and all polynomials $r(z), p(z)$ is essentially the virtual dimension 0 component of the Grassmannian Gr_1 defined in [16]. More precisely, Gr_1 consists of those W 's corresponding to the choices

$$W_1 := r(z) \mathcal{H}_+, \quad W_2 := \frac{1}{q(z)} \mathcal{H}_+, \quad (3.30)$$

for a polynomial $q(z)$ of degree k . But these are easily seen to be gauge equivalent to the choice (3.16). In fact, within gauge transformations, there is no loss of generality in choosing $r(z)$ as the monomial z^n ; i.e., choosing the subspaces W_1, W_2 as

$$W_1 = z^n \mathcal{H}_+, \quad W_2 = \frac{z^n}{p(z)} \mathcal{H}_+. \quad (3.31)$$

Note also that, from the viewpoint of the KP hierarchy, nothing new is added by considering W 's with virtual dimension different from 0, since there is always an equivalent τ -function in the zero charge sector. The charge sector only becomes relevant when considering lattices of τ -functions having the same group element \hat{g} in (2.29), with the lattice site given by the fermionic charge (or virtual dimension), thereby defining elements of an infinite flag manifold [3, 4].

3.2 KP τ -functions as finite determinants

Define the $n \times N$ matrix $A(B, D) \in \text{Mat}^{n \times N}$ by the following formula:

$$A_{(i\mu), (j\nu)}(B, D) := \frac{1}{2\pi i} \oint_{|z|=1} \frac{r_D(z)}{(z - \beta_j)^\nu (z - \delta_i)^\mu} dz, \quad (3.32)$$

$$1 \leq i \leq m, 1 \leq \mu \leq n_i, 1 \leq j \leq M, 1 \leq \nu \leq N_j.$$

and the further $n \times N$ matrix $A^0(B, D) \in \text{Mat}^{n \times N}$ by the following:

$$A_{(i\mu), (j\nu)}^0(B, D) := \binom{\mu + \nu - 2}{\nu - 1} \frac{(-1)^{\nu+1}}{(\beta_j - \delta_i)^{\mu+\nu-1}}. \quad (3.33)$$

Evaluating the integral using the Cauchy formula, it follows that these are related by right multiplication by the matrix $r_D(B)$:

Lemma 3.1

$$A(B, D) = A^0(B, D) r_D(B). \quad (3.34)$$

Proof. From the Cauchy residue formula and Leibniz' rule we have

$$\begin{aligned} A_{(i\mu), (j\nu)}(B, D) &= \frac{1}{(\nu - 1)!} \left. \frac{d^{\nu-1}}{dz^{\nu-1}} \right|_{z=\beta_j} \left(r_D(z) \frac{1}{(z - \delta_i)^\mu} \right) \\ &= \sum_{\xi=1}^{\nu} \frac{r_D^{(\nu-\xi)}(\beta_j)}{(\nu - \xi)!} \binom{\mu + \xi - 2}{\xi - 1} \frac{(-1)^{\xi-1}}{(\beta_j - \delta_i)^{\mu+\xi-1}} \\ &= \sum_{k=1}^M \sum_{\xi=1}^{\nu} A_{(i\mu), (k\xi)}^0(B, D) (r_D(B))_{(k\xi), (j\nu)}, \end{aligned} \quad (3.35)$$

since

$$(r_D(B))_{(k\xi),(j\nu)} = \delta_{jk} \frac{r_D^{(\nu-\xi)}(\beta_j)}{(\nu-\xi)!} \quad \text{for } \nu \geq \xi. \quad (3.36)$$

Q.E.D.

Denote by

$$A(B) := A(B, \Lambda_n), \quad (3.37)$$

the particular case where the matrix D is chosen as Λ_n . The matrix elements of $A(B)$ are easily computed to be

$$A_{a,(j\nu)}(B) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{z^{n-a}}{(z-\beta_j)^\nu} dz = \binom{n-a}{\nu-1} \beta_j^{n-\nu-a+1}, \quad (3.38)$$

$$1 \leq a \leq n, \quad 1 \leq j \leq M, \quad 1 \leq \nu \leq N_j$$

We also define the $n \times n$ matrix $K(D) \in \text{Mat}^{n \times n}$ with elements

$$K_{a,(i\mu)}(D) := \frac{1}{2\pi i} \oint_{z=\delta_i} \frac{z^{n-a}(z-\delta_i)^{\mu-1}}{r_D(z)} dz, \quad (3.39)$$

$$1 \leq a \leq n, \quad 1 \leq i \leq m, \quad 1 \leq \mu \leq n_i, \quad (3.40)$$

where the integral is over any counterclockwise contour containing only the pole $z = \delta_i$. The following gives the relation between these matrices

Lemma 3.2

$$A(B) = K(D)A(B, D). \quad (3.41)$$

Proof. Use Lagrange interpolation to express

$$z^{n-c} = \sum_{i,\mu} K_{c,(i\mu)}(D) \frac{r_D(z)}{(z-\delta_i)^\mu} \quad (3.42)$$

and substitute into (3.38).

Q.E.D.

Let $\mathbf{E}(B) \in \mathbf{C}^N$, $\mathbf{e}(D) \in \mathbf{C}^n$ and $\mathbf{k}(D) \in \mathbf{C}^n$ be the vectors:

$$\mathbf{E}(B)_{(j\nu)} = \delta_{\nu,1}, \quad 1 \leq j \leq M, \quad 1 \leq \nu \leq N_j, \quad (3.43)$$

$$\mathbf{e}(D)_{(i\mu)} = \delta_{\mu,1}, \quad 1 \leq i \leq m, \quad 1 \leq \mu \leq n_i, \quad (3.44)$$

$$\mathbf{k}_{(i\mu)}(D) = \frac{1}{2\pi i} \oint_{z=\delta_i} \frac{z^n(z-\delta_i)^{\mu-1}}{r_D(z)} dz. \quad (3.45)$$

We then have the following identities:

Lemma 3.3

$$\Lambda_n^T K(D) = K(D) D^T - \mathbf{e}(\Lambda_n) \mathbf{k}^T(D) \quad (3.46)$$

$$K(D) \mathbf{e}(D) = \mathbf{e}(\Lambda_n) \quad (3.47)$$

Proof. These follow from the definitions and the contour integral

$$\oint_{|z|=1} \frac{z^{n-c}}{r_D(z)} dz = \delta_{c1}, \quad 1 \leq c \leq n. \quad (3.48)$$

Q.E.D.

The following shows that the matrices $A^0(B, D)$ and $A(B, D)$ both satisfy the rank-1 condition (1.11) for the same pair of matrices B and D , but with different RHS.

Proposition 3.4

$$A(B)B - \Lambda_n^T A(B) = \mathbf{e}(\Lambda_n) \mathbf{E}(B)^T B^n \quad (3.49)$$

$$A^0(B, D)B - D^T A^0(B, D) = \mathbf{e}(D) \mathbf{E}(B)^T \quad (3.50)$$

$$A(B, D)B - D^T A(B, D) = \mathbf{e}(D) \mathbf{E}(B)^T r_D(B) \quad (3.51)$$

Proof. Eq. (3.49) follows from the definitions (and is also the special case of (3.51) when $D = \Lambda_n$). Eqs. (3.50) and (3.51) follow from substituting (3.34), (3.41), (3.46) and (3.47) into (3.49) and using the fact that B commutes with $r_D(B)$.

Eq. (3.50) can also be proved directly as follows. Since both B and D are in standard Jordan normal form, we may subdivide $A^0(B, D)$ into $m \times M$ blocks of sizes $n_i \times N_j$

$$A^0(B, D) = (\mathcal{A}_{ij}), \quad 1 \leq i \leq m, \quad 1 \leq j \leq M, \quad (3.52)$$

where each $n_i \times N_j$ block is of the form

$$\mathcal{A}_{\mu\nu}(\beta, \delta) = \binom{\mu + \nu - 2}{\nu - 1} \frac{(-1)^\nu}{(\beta - \delta)^{\mu + \nu - 1}} \quad (3.53)$$

with

$$\delta = \delta_i, \quad \beta = \beta_j, \quad 1 \leq i \leq m, \quad 1 \leq j \leq M, \quad 1 \leq \mu \leq n_i, \quad 1 \leq \nu \leq N_j. \quad (3.54)$$

Denoting the elementary Jordan blocks with matrix elements

$$J_{\mu\nu}(\beta) = \beta \delta_{\mu\nu} + \delta_{\mu+1, \nu}, \quad J_{\xi, \kappa}(\delta) = \delta \delta_{\xi, \kappa} + \delta_{\xi+1, \kappa}, \quad (3.55)$$

as $J(\beta)$ and $J(\delta)$ with the appropriate range of indices μ, ν, ξ, κ given by the dimension of the Jordan blocks, it is easily verified that

$$(\mathcal{A}(\beta, \delta)J_B - J_D^T(\delta)\mathcal{A}(\beta, \delta))_{\mu\nu} = \delta_{\mu 1}\delta_{\nu 1} \quad (3.56)$$

Applying this to each $n_i \times N_j$ block is equivalent to eq. (3.50).

Q.E.D.

Remark 3.3 In terms of the resolvents $(z\mathbf{I}_n - D^T)^{-1}$ and $(z\mathbf{I}_N - B)^{-1}$, (3.32) can be written equivalently as

$$A(B, D) = \frac{1}{2\pi i} \oint_{|z|=1} r_D(z)(z\mathbf{I}_n - D^T)^{-1} \mathbf{e}(D) \mathbf{E}(B)^T (z\mathbf{I}_N - B)^{-1} dz. \quad (3.57)$$

Also, note that

$$K(D)r_D(z)(z\mathbf{I}_n - D^T)^{-1} \mathbf{e}(D) = r_{\Lambda_n}(z)(z\mathbf{I}_n - \Lambda_n^T)^{-1} \mathbf{e}(\Lambda_n), \quad (3.58)$$

which explains (3.41) through (3.57).

The τ -function $\tau_{W(B, C, D)}(\mathbf{t})$ corresponding to the element $W(B, C, D) \in \text{Gr}_{\mathcal{H}_+^0}(\mathcal{H})$ defined in (3.28), with W_1, W_2 defined in (3.16), is given by the following:

Theorem 3.5

$$\tau_{W(B, C, D)}(\mathbf{t}) = \det(A(B, D)e^{\sum_{i=1}^{\infty} t_i B^i} C^T) \quad (3.59)$$

$$= \det(A^0(B, D)e^{\sum_{i=1}^{\infty} t_i B^i} r_D(B) C^T) \quad (3.60)$$

$$= \kappa(D)^{-1} \det(A(B)e^{\sum_{i=1}^{\infty} t_i B^i} C^T), \quad (3.61)$$

where

$$\kappa(D) := \det(K(D)). \quad (3.62)$$

Proof. We begin with the case $D = \Lambda_n$, $r_D(z) = z^n$ and B of the diagonal form (1.23); that is, all Jordan blocks of B have dimension $N_j = 1$. For this case, a basis for the space $W(B, C, \Lambda_n)$ may be taken as

$$\{q_a, z^{n+i}\}_{a=1, \dots, n}, \quad i \in \mathbf{N}, \quad (3.63)$$

where

$$q_a := \sum_{j=1}^N C_{a,j} \frac{z^n}{z - \beta_j} \quad (3.64)$$

Since the subspace $z^n \mathcal{H}_+ \subset W$ is invariant under the Γ_+ action, we may choose

$$\{q_a(\mathbf{t}), z^{n+i}\}_{a=1,\dots,n}, \quad i \in \mathbf{N} \quad (3.65)$$

as basis for the orbit space

$$W(\mathbf{t}) := \gamma_+(\mathbf{t})W, \quad (3.66)$$

where

$$\begin{aligned} q_a(\mathbf{t}) &:= \sum_{j=1}^N \frac{C_{a,j} e^{\sum_{i=1}^{\infty} t_i \beta_j^i} z^n}{z - \beta_j} \\ &= \sum_{j=1}^N \frac{C_{a,j}(\mathbf{t}) z^n}{z - \beta_j}. \end{aligned} \quad (3.67)$$

Here $\{C_{a,j}(\mathbf{t})\}$ are the matrix elements of

$$C(\mathbf{t}) := C e^{\sum_{i=1}^{\infty} t_i B^i} \quad (3.68)$$

The τ -function $\tau_{W(B,C,\Lambda_n)}(\mathbf{t})$ is the determinant of the orthogonal projection to \mathcal{H}_+ ; that is, of the infinite matrix \mathcal{M} of scalar products of the basis elements $\{z^i\}_{i \in \mathbf{N}}$ of \mathcal{H}_+ with those of $W(B, C, \Lambda_n)$. We have, for $1 \leq i, a \leq n$,

$$\mathcal{M}_{i,a} = (z^{i-1}, q_a(\mathbf{t})) = \frac{1}{2\pi i} \oint_{|z|=1} dz \sum_{j=1}^N C_{a,j}(\mathbf{t}) \frac{z^{n-i}}{z - \beta_j} = \sum_{j=1}^N C_{a,j}(\mathbf{t}) \beta_j^{n-i} \quad (3.69)$$

and

$$\mathcal{M}_{n+i,j} = \mathcal{M}_{j,n+i} = \delta_{n+i,j}, \quad i, j \in \mathbf{N}^+. \quad (3.70)$$

Since \mathcal{M} is block triangular, only its first $N \times N$ block contributes to its determinant, and therefore

$$\tau_{W(B,C,\Lambda_n)}(\mathbf{t}) = \det(\mathcal{M}) = \det \left(A(B) e^{\sum_{i=1}^{\infty} t_i B^i} C^T \right), \quad (3.71)$$

where $A(B) = V_{n,N}(\beta)$ is the truncated Vandermonde matrix defined in (1.24).

The case where B has general Jordan blocks of dimensions $\{N_j\}_{j=1,\dots,M}$ and $D = \Lambda_n$ is proved in exactly the same way. The basis elements $q_a(\mathbf{t})$ are replaced by

$$q_a(\mathbf{t}) := \sum_{j=1}^M \sum_{\nu=1}^{N_j} C_{a,(j,\nu)}(\mathbf{t}) \frac{z^n}{(z - \beta_j)^\nu}, \quad (3.72)$$

where $C_{a,(j,\nu)}(\mathbf{t})$ are the matrix elements of

$$C(\mathbf{t}) = Ce^{\sum_{i=1}^{\infty} t_i (B^T)^i} \quad (3.73)$$

and B is the $N \times N$ matrix of general nondegenerate Jordan form defined in eq. (1.52). The residue calculation is carried out in the same way, with the higher order poles giving derivatives of the columns of the truncated Vandermonde matrix (1.24) to the same order as the pole. The resulting form for the matrix A is the generalized truncated Vandermonde matrix $V'_{n,N}$ with elements

$$A_{a,(j,\nu)} = \frac{(n-a+1)!}{(n-a-\nu+1)!} (\beta_j)^{n-a-\nu+1}, \quad (3.74)$$

which is the matrix $A(B, D)$ in eq (3.32) for the special case $m = 1$, $\delta_1 = 0$.

To prove the general case, where both B and D have general Jordan block structure (1.52) and (1.54), we proceed in the same way, but replace the monomials $\{z^i\}_{i=0,\dots,n-1}$ spanning the orthogonal complement to $z^n \mathcal{H}_+$ in \mathcal{H}_+ by the elements $\{d_{i\mu}(z)\}$ defined in (3.19) that span the orthogonal complement to $W_1 = r(z) \mathcal{H}_+$ and the basis elements spanning W/W_1 and $W(\mathbf{t})/W_1$ by

$$q_{i\mu} := \sum_{j=1}^M \sum_{\nu=1}^{N_j} C_{i\mu,j\nu} \frac{r_D(z)}{(z - \beta_j)^\nu}, \quad 1 \leq i \leq m, \quad 1 \leq \mu \leq n_i. \quad (3.75)$$

and

$$q_{i\mu}(\mathbf{t}) := \sum_{j=1}^M \sum_{\nu=1}^{N_j} C_{i\mu,j\nu}(\mathbf{t}) \frac{r_D(z)}{(z - \beta_j)^\nu}, \quad 1 \leq i \leq m, \quad 1 \leq \mu \leq n_i. \quad (3.76)$$

Computation of the determinant of the orthogonal projection leads to the evaluation of the contour integrals

$$\frac{1}{2\pi i} \oint dz \frac{r_D(z)}{(z - \delta_i)^\mu (z - \beta_j)^\nu} = A_{(i\mu),(j\nu)}(B, D), \quad (3.77)$$

resulting in the general form (3.59) and (3.61). Equalities (3.60) and (3.61) then follow from (3.34) and (3.41). **Q.E.D.**

Remark 3.4 The expressions (3.59)-(3.61) can be rewritten equivalently as

$$\tau_{W(B,C,D)} = \det(P(D))^{-1} \det(\tilde{A} e^{\sum_{i=1}^{\infty} t_i B^i} \tilde{C}^T) \quad (3.78)$$

where

$$\tilde{C}^T = (R(B)^T)^{-1} C^T, \quad \tilde{A} = P(D)^T A(B, D) R(B) \quad (3.79)$$

and $P(D) \in GL(n)$ and $R(B) \in GL(N)$ are arbitrary elements of the stabilizers of D and B , respectively, under conjugation. The rank-1 condition satisfied by the resulting matrix \tilde{A} is then

$$\tilde{A}B - D^T \tilde{A} = \mathbf{f}\mathbf{g}^T \quad (3.80)$$

where

$$\mathbf{f} = P(D)^T \mathbf{e}(D), \quad \mathbf{g} = R(B) \mathbf{E}(B) \quad (3.81)$$

are now arbitrary vectors determined by a suitable choice of $P(D)$, $R(B)$, satisfying the generic conditions

$$f_{i,1} \neq 0, \quad g_{j,1} \neq 0, \quad 1 \leq i = 1 \leq m, \quad 1 \leq j \leq M. \quad (3.82)$$

Those cases with degeneracies violating (3.82) will be dealt with in a subsequent publication.

The solutions of this type are “generic”, in the sense that they form an open dense set defined by the requirements that the eigenvalues $(\delta_1, \dots, \delta_m)$ and $(\beta_1, \dots, \beta_M)$ be distinct and the conditions (3.82) be satisfied. In the sequel [2], a complete classification of all solutions of the rank-1 condition (1.13), will given, for B and D having arbitrary Jordan forms, within equivalence under the action of their stabilizers.

Remark 3.5 Eq. (3.41) shows that the choice of the matrix D is in fact immaterial, since it only affects the τ -function by a constant multiplicative factor. This is a little surprising, since a change in D is equivalent to a change in $r(z)$, and hence is actually a gauge transformation that should give rise to a multiplicative linear exponential factor. The absence of this factor is due to the fact that an “admissible section” must be chosen in the dual determinantal line bundle (cf. [16]) when considering the lift of the Γ_{\pm} actions from the Grassmannian. From this it follows that, whereas the two abelian subgroups Γ_+ and Γ_- , when acting upon the Hilbert space \mathcal{H} mutually commute, the actions induced on the holomorphic sections of the dual determinantal line bundle over the Grassmannian, only commute within a scalar multiplicative factor; i.e., there is a central extension.

To obtain this in the finite Grassmannian approach, we recall that the notion of the determinant of the projection map $W(\mathbf{t}) \rightarrow \mathcal{H}_+$ is only well defined if a basis is chosen for both $W(\mathbf{t})$ and \mathcal{H}_+ allowing us to view this as an endomorphism. In the proof of (3.59), we have chosen the basis $\{r_D(z)\mathcal{H}_+z^j, q_{i\mu}\}_{j \in \mathbf{N}, 1 \leq i \leq m, 1 \leq \mu \leq n_i}$ for $W(B, C, D)$ and $\{r_D(z)\mathcal{H}_+z^j, d_{i\mu}\}_{j \in \mathbf{N}, 1 \leq i \leq m, 1 \leq \mu \leq n_i}$ for \mathcal{H}_+ . To obtain the missing gauge factor $e^{-\sum_{i=1}^{\infty} t_i \text{tr}(D)^i}$, as in eq. (3.90), Theorem 3.7 below, we may choose instead the time dependent basis $\{r_D(z)\mathcal{H}_+z^j, \gamma_+(\mathbf{t})d_{i\mu}(z)\}_{j \in \mathbf{N}, 1 \leq i \leq m, 1 \leq \mu \leq n_i}$ for \mathcal{H}_+ . The corresponding calculation is made in the next subsection using the fermionic representation, in which this gauge factor appears automatically, since the fermionic Fock space is precisely the space of admissible holomorphic sections of the dual determinantal line bundle.

3.3 Fermionic representation

We now give an alternative derivation of formula (3.59) using fermionic operators. Define the following Fermi creation operators

$$w_{(i,\mu)}(B, C, D) := \sum_{j=1}^M \sum_{\nu=1}^{N_j} C_{(i,\mu),(j,\nu)} \Psi_D^\nu(\beta_j), \quad (3.83)$$

$$i = 1, \dots, m, \quad \mu = 1, \dots, n_i,$$

where

$$\Psi_D^\nu(\beta_j) := \frac{1}{2\pi i} \oint_{z=\beta_j} \frac{r_D(z)\psi(z)}{(z-\beta_j)^\nu} dz = \frac{1}{(\nu-1)!} \frac{\partial^\nu}{\partial \beta_j^\nu} (\psi(\beta_j) r_D(\beta_j)), \quad (3.84)$$

with the integral taken over a small contour containing only the pole at $z = \beta_j$. Similarly, define the annihilation operators

$$(\Psi^\mu(\delta_i))^\dagger := \frac{1}{2\pi i} \oint_{z=\delta_i} \frac{\psi^\dagger(z)}{(z-\delta_i)^\mu} dz = \frac{1}{(\mu-1)!} \frac{\partial^{\mu-1}}{\partial \delta_j^{\mu-1}} \psi(\delta_j). \quad (3.85)$$

Lemma 3.6 *The images of the subspaces W_1 , W_2 and W under the Plücker map (2.14), are*

$$\mathfrak{Pl}(W_1) = \prod_{i=1}^m \prod_{\mu=1}^{n_i} (\Psi^\mu(\delta_i))^\dagger |0\rangle, \quad (3.86)$$

$$\mathfrak{Pl}(W_2) = \prod_{j=1}^M \prod_{\nu=1}^{N_j} \Psi_D^\nu(\beta_j) \prod_{i=1}^m \prod_{\mu=1}^{n_i} (\Psi^\mu(\delta_i))^\dagger |0\rangle, \quad (3.87)$$

$$\mathfrak{Pl}(W) = \prod_{i=1}^m \prod_{\nu=1}^{n_j} w_{(j,\nu)}(B, C, D) \prod_{i=1}^m \prod_{\mu=1}^{n_i} (\Psi^\mu(\delta_i))^\dagger |0\rangle. \quad (3.88)$$

Proof. We begin with (3.86). Viewing $\{\psi_j^\dagger \sim e^j\}$ as elements of the dual space \mathcal{H}^* , acting on \mathcal{F} through the Clifford representation by inner products, as in eq. (2.22), a basis for the annihilator of $W_1 \subset \mathcal{H}_+$ consists of $\{(\Psi^\mu(\delta_i))^\dagger\}_{i=1,\dots,d; \mu=1,\dots,n_i}$. The latter follows from the fact that $(\Psi^\mu(\delta))^\dagger$, as a linear form on \mathcal{H} , acting upon an element $f(z) \in \mathcal{H}_+$, under the identification $z^i \sim e_{-i-1}$, evaluates to its derivative at $z = \delta$:

$$(\Psi^\mu(\delta))^\dagger(f(z)) = \frac{1}{(\mu-1)!} f^{(\mu-1)}(\delta). \quad (3.89)$$

Eq. (3.86) follows, since the image $\mathfrak{Pl}(U)$, of any subspace $U \subset \mathcal{H}_+$ under the Plücker map $\mathfrak{Pl} : U \rightarrow \mathcal{F}$ is the joint kernel of the elements of its annihilator within the Clifford representation which, for $U = W_1$, is given by the r.h.s. of eq. (3.86).

To prove (3.87), we note that the subspace W_2 is obtained by extending W_1 by the basis elements $\{q_{j\nu}\}_{1 \leq j \leq M, 1 \leq \nu \leq N_j}$. The Plücker image of W_2 is therefore obtained by applying the wedge product of the elements $\{q_{j\nu}\}$ to the Plücker image (3.86) of W_1 . But this is equivalent to applying the product of the operators $\Psi_D^\nu(\beta_j)$, since each of these may be expressed as the exterior product with $q_{j\nu-1}$ plus a linear combination of lower order terms $\{q_{j,\nu-i}\}$, $i = 2, \dots, \nu$. Its Plücker image is therefore given by (3.87).

Finally, to obtain (3.88) as the Plücker image of W , we replace the product of creation operators $\prod_{\nu=1}^{N_j} \Psi_D^\nu(\beta_j)$ in (3.87) by the product $\prod_{i=1}^m \prod_{\mu=1}^{n_i} (\Psi^\mu(\delta_i))^\dagger$ corresponding to the basis elements $\{q_{i\mu}\}_{1 \leq i \leq m, 1 \leq \mu \leq n_i}$ that complete the basis for W . **Q.E.D.**

From this lemma and the equivariance of the Plücker map, it follows that the τ -function of Theorem 3.5, eq. (3.59), may equivalently be expressed in fermionic form as:

Theorem 3.7

$$\begin{aligned} \langle 0 | \hat{\gamma}_+(\mathbf{t}) \prod_{j=1}^m \prod_{\nu=1}^{N_j} w_{(j,\nu)}(B, C) \prod_{i=1}^m \prod_{\mu=1}^{n_i} (\Psi^\mu(\delta_i))^\dagger | 0 \rangle &= e^{-\sum_{i=1}^\infty t_i \text{tr}(D)^i} \tau_{W(B,C,D)}(\mathbf{t}) \\ &= e^{-\sum_{i=1}^\infty t_i \text{tr}(D)^i} \det \left(A(B, D) e^{\sum_{i=1}^\infty t_i B^i} C^T \right). \end{aligned} \quad (3.90)$$

Proof. The proof is based on Wick's identity, starting from the fermionic expression (3.90). It follows from Lemma 3.6 that $\tau_{W(B,C,D)}(\mathbf{t})$ is given by

$$\tau_{W(B,C,D)}(\mathbf{t}) = \langle 0 | \hat{\gamma}_+(\mathbf{t}) \mathfrak{Pl}(W) \rangle = \langle 0 | \hat{\gamma}_+(\mathbf{t}) \prod_{i=1}^m \prod_{\mu=1}^{n_i} (\Psi^\mu(\delta_i))^\dagger \prod_{j=1}^m \prod_{\nu=1}^{n_j} w_{(j,\nu)}(B, C) | 0 \rangle. \quad (3.91)$$

We first note that, since the group Γ_+ stabilizes the vacuum, left multiplication by the element $\hat{\gamma}_+(\mathbf{t})$ in (3.91) is equivalent to conjugation of all the Fermi creation and annihilation operators:

$$\begin{aligned} (\Psi^\mu(\delta_i))^\dagger \rightarrow \hat{\gamma}_+(\mathbf{t}) (\Psi^\mu(\delta_i))^\dagger \hat{\gamma}_+^{-1}(\mathbf{t}) &= \frac{1}{(\mu-1)!} \frac{\partial^{\mu-1}}{\partial \delta_i^{\mu-1}} \left(e^{-\sum_{k=1}^\infty t_k \delta_i^k} \psi(\delta_i)^\dagger \right), \\ \Psi_D^\nu(\beta_j) \rightarrow \hat{\gamma}_+(\mathbf{t}) \Psi_D^\nu(\beta_j) \hat{\gamma}_+^{-1}(\mathbf{t}) &= \frac{1}{(\nu-1)!} \frac{\partial^{\nu-1}}{\partial \beta_j^{\nu-1}} \left(e^{\sum_{k=1}^\infty t_k \beta_j^k} \psi(\beta_j) r_D(\beta_j) \right). \end{aligned} \quad (3.92)$$

The net effect is to multiply the vacuum expectation value of the terms without the $\hat{\gamma}_+(\mathbf{t})$ factor by an overall linear exponential factor $e^{-\sum_{k=1}^{\infty} t_k \text{tr}(D^k)}$ and replace the matrix C in the expression (3.83) for $w_{(i,\mu)}(B, C, D)$ by $C(\mathbf{t})$, as in eq. (3.68).

$$\tau_{W(B,C,D)}(\mathbf{t}) = e^{-\sum_{k=1}^{\infty} t_k \text{tr}(D^k)} \langle 0 | \prod_{i=1}^m \prod_{\mu=1}^{n_i} (\Psi^\mu(\delta_i))^\dagger \prod_{j=1}^m \prod_{\nu=1}^{n_j} w_{(j,\nu)}(B, C(\mathbf{t})) | 0 \rangle. \quad (3.93)$$

By Wick's identity, the vacuum matrix element can be written as an $n \times n$ determinant

$$\begin{aligned} \langle 0 | \prod_{i=1}^m \prod_{\mu=1}^{n_i} (\Psi^\mu(\delta_i))^\dagger \prod_{j=1}^m \prod_{\nu=1}^{n_j} w_{(j,\nu)}(B, C(\mathbf{t})) | 0 \rangle &= \det \langle 0 | (\Psi^\mu(\delta_i))^\dagger w_{(j,\nu)}(B, C(\mathbf{t})) | 0 \rangle \\ &= \det(A(B, D)C^T(\mathbf{t})), \end{aligned} \quad (3.94)$$

since

$$\begin{aligned} \langle 0 | \Psi^\nu(\beta_j) (\Psi^\mu(\delta_i))^\dagger | 0 \rangle &= \frac{1}{2\pi i} \frac{1}{(\mu-1)!} \frac{\partial^{\mu-1}}{\partial \delta_i^{\mu-1}} \oint_{z=\beta_j} \frac{\langle 0 | \psi(z) \psi^\dagger(\delta_i) | 0 \rangle r_D(z)}{(z-\beta_j)^\nu} dz \\ &= \frac{1}{2\pi i} \frac{1}{(\mu-1)!} \frac{\partial^{\mu-1}}{\partial \delta_i^{\mu-1}} \oint_{z=\beta_j} \frac{r_D(z)}{(z-\beta_j)^\nu (z-\delta_i)^\mu} dz \\ &= \frac{1}{2\pi i} \oint_{z=\beta_j} \frac{r_D(z)}{(z-\beta_j)^\nu (z-\delta_i)^\mu} dz \\ &= A_{(i\mu), (j\nu)}(B, D). \end{aligned} \quad (3.95)$$

Q.E.D.

A slightly more general class of τ -functions having a finite dimensional exponential determinantal form may be constructed as follows. For three positive integers l, n, N with $l \leq n, l \leq N$, but no restriction relating n and N , we again choose a pair $D \in \text{Mat}^{n \times n}$, $B \in \text{Mat}^{N \times N}$ of square matrices, and a rectangular matrix $A \in \text{Mat}^{n \times N}$ satisfying the rank-1 condition (1.13) for a pair of vectors $\mathbf{f} \in \mathbf{C}^n$, $\mathbf{g} \in \mathbf{C}^N$. Then choosing any pair $F \in \text{Mat}^{l \times n}$, $C \in \text{Mat}^{l \times N}$ such that FAC^T is invertible, the following $l \times l$ determinantal formula

$$\tau_{(A,B,C,D,F)}^f(\mathbf{t}) := \det(F e^{-\sum_{i=1}^{\infty} t_i (D^T)^i} A e^{\sum_{j=1}^{\infty} t_j B^j} C^T) \quad (3.96)$$

is shown to define a KP τ -function in the next section. The appendix gives a simple direct verification that $\tau_{(A,B,C,D,F)}^f(\mathbf{t})$ satisfies the Hirota bilinear relations.

Assuming the eigenvalues of B and D to be mutually distinct, $\tau_{(A,B,C,D,F)}^f(\mathbf{t})$ may also be expressed in fermionic operator form as

$$\tau_{(\tilde{A}, B, \tilde{C}, D, \tilde{F})}^f(\mathbf{t}) = \langle 0 | \hat{\gamma}_+(\mathbf{t}) \prod_{a=1}^l w_a(B, C) \prod_{b=1}^l v_b^\dagger(F, D) | 0 \rangle, \quad (3.97)$$

where \tilde{A} , \tilde{C} are defined as in eq. (3.79), with \tilde{A} satisfying the rank-1 condition (3.80) for \mathbf{f} , \mathbf{g} defined in eq. (3.81), \tilde{F} defined by

$$\tilde{F} = F(\tilde{K}^T(D))^{-1}, \quad (3.98)$$

and the fermionic creation and annihilation operators w_a , v_a^\dagger are defined by

$$w_a(B, C, D) := \sum_{j=1}^M \sum_{\nu=1}^{N_j} C_{a,(j,\nu)} \Psi_D^\nu(\beta_j), \quad (3.99)$$

$$v_b^\dagger(F, D) := \sum_{i=1}^m \sum_{\mu=1}^{n_i} F_{b,(i,\mu)} (\Psi^\mu(\delta_i))^\dagger, \quad a = 1, \dots, l. \quad (3.100)$$

This may be shown using Wick's identity, exactly as in the proof of Theorem 3.7. Choosing $l = n$ and $\det F = 1$, the case of eq. (3.90) is recovered.

4 Affine coordinates and Schur function expansions

The aim of this section is to find the affine coordinates of the subspace $W \in Gr_{\mathcal{H}_+}(\mathcal{H})$ corresponding to a given generalized Gekhtman-Kasman τ -function $\tau_{(A,B,C,D,F)}^f(\mathbf{t})$ as defined in (1.15). This allows us to obtain the Plücker coordinates of W through the Giambelli formula (2.21) and hence the Schur function expansion of $\tau_{(A,B,C,D,F)}^f(\mathbf{t})$. In particular, this gives the Schur function expansion for τ -functions of the original Gekhtman-Kasman form $\tau_{(A,B,C)}^f(\mathbf{t})$, up to an explicit gauge factor.

4.1 Subspaces with geometric affine coordinates

Let n and N be positive integers (with no assumption on their relative size) and consider a quintuple $(\mathbf{f}, \mathbf{g}, B, D, M)$ consisting of a pair of vectors

$$(\mathbf{f}, \mathbf{g}) \in \mathbf{C}^n \times \mathbf{C}^N, \quad (4.1)$$

a pair of square matrices

$$B \in \text{Mat}^{N \times N}, \quad D \in \text{Mat}^{n \times n} \quad (4.2)$$

whose eigenvalues are inside the unit disk in the complex plane, and a rectangular matrix

$$M \in \text{Mat}^{N \times n}. \quad (4.3)$$

Consider the subspace $W(\mathbf{f}, \mathbf{g}, B, D, M)$ in the big cell of the Segal-Wilson Grassmannian defined by choosing the affine coordinates to have the special form

$$\mathcal{A}_{ij} = \mathbf{g}^T B^j M (D^T)^i \mathbf{f} \quad i, j = 0, 1, \dots \quad (4.4)$$

This will be referred to as *geometric affine coordinates*, since the dependence of \mathcal{A}_{ij} on D and B are given through the matrix-valued “geometric sequences” $\{(D^T)^i\}_{i \in \mathbf{N}}$ and $\{B^j\}_{j \in \mathbf{N}}$. That is,

$$W(\mathbf{f}, \mathbf{g}, B, D, M) = \text{span} \left\{ z^i + \sum_{j=0}^{\infty} (\mathbf{g}^T B^j M (D^T)^i \mathbf{f}) z^{-j-1} \right\}_{i \in \mathbf{N}}. \quad (4.5)$$

Since the spectrum of B is contained in the unit disk, the basis elements can be rewritten in terms of the resolvent of B as

$$z^i + \mathbf{g}^T (z \mathbf{I}_N - B)^{-1} M (D^T)^i \mathbf{f}, \quad i = 0, 1, \dots \quad (4.6)$$

Therefore $W(\mathbf{f}, \mathbf{g}, B, D, M)$ is the graph of the operator

$$\begin{aligned} T : \mathcal{H}_+ &\rightarrow \mathcal{H}_- \\ \phi &\mapsto \frac{1}{2\pi i} \oint_{|\zeta|=1} [\mathbf{g}^T (z \mathbf{I}_N - B)^{-1} M (\zeta \mathbf{I}_n - D^T)^{-1} \mathbf{f}] \phi(\zeta) d\zeta. \end{aligned} \quad (4.7)$$

Theorem 4.1 *We have the inclusions*

$$\mu_D(z) \mathcal{H}_+ \subset W(\mathbf{f}, \mathbf{g}, B, D, M) \subset \frac{1}{\mu_B(z)} \mathcal{H}_+, \quad (4.8)$$

where μ_D and μ_B are the minimal polynomials of the matrices D and B , respectively.

Proof. For any $\phi(z) \in \mathcal{H}_+$ the function $[T\phi](z)$ is rational with possible poles only at the eigenvalues of B . The common denominator of the entries of the resolvent matrix $(z \mathbf{I}_N - B)^{-1}$ is equal to the minimal polynomial $\mu_B(z)$ and therefore

$$[T\phi](z) \in \frac{1}{\mu_B(z)} \mathcal{H}_+ \quad \text{for all } \phi(z) \in \mathcal{H}_+. \quad (4.9)$$

On the other hand, if $p(z)$ is a polynomial divisible by the minimal polynomial $\mu_D(z)$, we have

$$\mu_D(z) | p(z) \quad \Rightarrow \quad [Tp](z) = \mathbf{g}^T (z \mathbf{I}_N - B)^{-1} M p(D^T) \mathbf{f} = 0. \quad (4.10)$$

By continuity,

$$T(\mu_D(z)\phi(z)) = 0 \quad \text{for all } \phi(z) \in \mathcal{H}_+, \quad (4.11)$$

since the polynomials form a dense subset in \mathcal{H}_+ . This clearly implies that

$$\mu_D(z)\mathcal{H}_+ \subset W(\mathbf{f}, \mathbf{g}, B, D, M). \quad (4.12)$$

Q.E.D.

The chain of inclusions (4.8) evidently also implies the weaker one

$$r_D(z)\mathcal{H}_+ \subset W(\mathbf{f}, \mathbf{g}, B, D, M) \subset \frac{1}{r_B(z)}\mathcal{H}_+, \quad (4.13)$$

where $r_D(z)$ and $r_B(z)$ stand for the characteristic polynomials of the matrices D and B , respectively.

Theorem 4.2 *The τ -function of $W(\mathbf{f}, \mathbf{g}, B, D, M)$ is given by*

$$\begin{aligned} & \tau_{W(\mathbf{f}, \mathbf{g}, B, D, M)}(\mathbf{t}) \\ &= \det \left(\mathbf{I}_n + e^{-\sum t_i (D^T)^i} \left(\frac{1}{2\pi i} \oint_{|z|=1} (z \mathbf{I}_n - D^T)^{-1} \mathbf{f} \mathbf{g}^T (z \mathbf{I}_N - B)^{-1} e^{\sum t_i z^i} dz \right) M \right)_{n \times n} \end{aligned} \quad (4.14)$$

$$= \det \left(\mathbf{I}_N + M e^{-\sum t_i (D^T)^i} \left(\frac{1}{2\pi i} \oint_{|z|=1} (z \mathbf{I}_n - D^T)^{-1} \mathbf{f} \mathbf{g}^T (z \mathbf{I}_N - B)^{-1} e^{\sum t_i z^i} dz \right) \right)_{N \times N}. \quad (4.15)$$

Proof. Consider the block decomposition

$$\gamma_+(\mathbf{t}) = \begin{pmatrix} a(\mathbf{t}) & b(\mathbf{t}) \\ 0 & d(\mathbf{t}) \end{pmatrix} \quad (4.16)$$

with respect to the splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Following [16], the τ -function of $W(\mathbf{f}, \mathbf{g}, B, D, M)$ is given by

$$\tau_{W(\mathbf{f}, \mathbf{g}, B, D, M)}(\mathbf{t}) = \det (\mathbf{I}_{\mathcal{H}_+} + T(\mathbf{t})), \quad (4.17)$$

where

$$T(\mathbf{t}) : \mathcal{H}_+ \rightarrow \mathcal{H}_+, \quad T(\mathbf{t}) := b(\mathbf{t}) T a(\mathbf{t})^{-1}. \quad (4.18)$$

Assuming $\phi \in \mathcal{H}_+$,

$$\begin{aligned} [T(\mathbf{t})\phi](z) &= \mathbf{g}^T \left[\frac{1}{2\pi i} \oint_{|\zeta|=1} e^{\sum t_i \zeta^i} (\zeta \mathbf{I}_N - B)^{-1} \frac{d\zeta}{\zeta - z} \right] M e^{-\sum t_i (D^T)^i} \phi(D^T) \mathbf{f} \\ &= \mathbf{g}^T \left(e^{\sum t_i z^i} \mathbf{I}_N - e^{\sum t_i B^i} \right) (z \mathbf{I}_N - B)^{-1} M e^{-\sum t_i (D^T)^i} \phi(D^T) \mathbf{f}, \end{aligned} \quad (4.19)$$

given that all the eigenvalues of B are inside the unit circle. Therefore the operator $T(\mathbf{t})$ may be written in factorized form as

$$T(\mathbf{t}) = L_2(\mathbf{t})L_1(\mathbf{t}) \quad (4.20)$$

where

$$\begin{aligned} L_1(\mathbf{t}) : \mathcal{H}_+ &\rightarrow \mathbf{C}^n \\ \phi &\mapsto e^{-\sum t_i (D^T)^i} \phi(D^T) \mathbf{f}, \end{aligned} \quad (4.21)$$

and

$$\begin{aligned} L_2(\mathbf{t}) : \mathbf{C}^n &\rightarrow \mathcal{H}_+ \\ \mathbf{v} &\mapsto \mathbf{g}^T \left(e^{\sum t_i z^i} \mathbf{I}_N - e^{\sum_{i=1}^\infty t_i B^i} \right) (z \mathbf{I}_N - B)^{-1} M \mathbf{v}. \end{aligned} \quad (4.22)$$

By inverting the order of the operators $L_1(\mathbf{t})$ and $L_2(\mathbf{t})$ we obtain the map

$$\begin{aligned} L_1(\mathbf{t})L_2(\mathbf{t}) : \mathbf{C}^n &\rightarrow \mathbf{C}^n \\ \mathbf{v} &\mapsto e^{-\sum t_i (D^T)^i} \left[\frac{1}{2\pi i} \oint_{|z|=1} (z \mathbf{I}_n - D^T)^{-1} \mathbf{f} \mathbf{g}^T (z \mathbf{I}_N - B)^{-1} e^{\sum t_i z^i} dz \right] M \mathbf{v}, \end{aligned} \quad (4.23)$$

where the second term vanishes since the fact that the eigenvalues of D and B are inside the unit circle implies that

$$\frac{1}{2\pi i} \oint_{|z|=1} (z \mathbf{I}_n - D^T)^{-1} \mathbf{f} \mathbf{g}^T (z \mathbf{I}_N - B)^{-1} dz = 0. \quad (4.24)$$

Applying the Weinstein–Aronszajn identity,

$$\begin{aligned} \tau_{W(\mathbf{f}, \mathbf{g}, B, D, M)}(t) &= \det (\mathbf{I}_{\mathcal{H}_+} + L_2(\mathbf{t})L_1(\mathbf{t})) \\ &= \det (\mathbf{I}_n + L_1(\mathbf{t})L_2(\mathbf{t})) \\ &= \det \left(\mathbf{I}_n + e^{-\sum t_i (D^T)^i} \left(\frac{1}{2\pi i} \oint_{|z|=1} (z \mathbf{I}_n - D^T)^{-1} \mathbf{f} \mathbf{g}^T (z \mathbf{I}_N - B)^{-1} e^{\sum t_i z^i} dz \right) M \right) \\ &= \det \left(\mathbf{I}_N + M e^{-\sum t_i (D^T)^i} \left(\frac{1}{2\pi i} \oint_{|z|=1} (z \mathbf{I}_n - D^T)^{-1} \mathbf{f} \mathbf{g}^T (z \mathbf{I}_N - B)^{-1} e^{\sum t_i z^i} dz \right) \right) \end{aligned} \quad (4.25)$$

Q.E.D.

The generalized Giambelli formula (2.21) implies that the Plücker coordinates associated to $W(\mathbf{f}, \mathbf{g}, B, D, M)$ are

$$\pi_{(a_1, \dots, a_k | b_1, \dots, b_k)}(W(\mathbf{f}, \mathbf{g}, B, D, M)) = (-1)^{\sum_{i=1}^k b_i} \det (\mathbf{g}^T B^{b_j} M (D^T)^{a_i} \mathbf{f})_{i,j=1}^k, \quad (4.26)$$

Therefore the τ -function $\tau_{W(\mathbf{f}, \mathbf{g}, B, D, M)}(\mathbf{t})$ has following Schur function expansion.

Corollary 4.3

$$\tau_{W(\mathbf{f}, \mathbf{g}, B, D, M)}(\mathbf{t}) = \sum_{(\mathbf{a}|\mathbf{b})} \det(\mathbf{g}^T(-B)^{b_j} M(D^T)^{a_i} \mathbf{f})_{i,j=1}^k S_{(\mathbf{a}|\mathbf{b})}(\mathbf{t}). \quad (4.27)$$

4.2 Specialization to $\tau_{(A,B,C,D,F)}^f(\mathbf{t})$

We now specialize the above to the subspaces corresponding to τ -functions of the generalized Gekhtman-Kasman form $\tau_{(A,B,C,D,F)}^f(\mathbf{t})$ as defined in eq. (1.15).

We first note that the rank-1 condition

$$AB - D^T A = \mathbf{f} \mathbf{g}^T \quad (4.28)$$

is equivalent to the resolvent identity

$$A(z \mathbf{I}_N - B)^{-1} = (z \mathbf{I}_n - D^T)^{-1} A + (z \mathbf{I}_n - D^T)^{-1} \mathbf{f} \mathbf{g}^T (z \mathbf{I}_N - B)^{-1}. \quad (4.29)$$

More generally, equation (4.28) implies that for any function $\phi(z)$ holomorphic in the unit disk we have

$$A\phi(B) = \phi(D^T)A + \frac{1}{2\pi i} \oint_{|\zeta|=1} (\zeta \mathbf{I}_n - D^T)^{-1} \mathbf{f} \mathbf{g}^T (\zeta \mathbf{I}_N - B)^{-1} \phi(\zeta) d\zeta. \quad (4.30)$$

In particular,

$$\frac{1}{2\pi i} \oint_{|\zeta|=1} (\zeta \mathbf{I}_n - D^T)^{-1} \mathbf{f} \mathbf{g}^T (\zeta \mathbf{I}_N - B)^{-1} e^{\sum t_i \zeta^i} d\zeta = A e^{\sum t_i B^i} - e^{\sum t_i (D^T)^i} A, \quad (4.31)$$

for any $e^{\sum t_i z^i} \in \Gamma_+$.

Theorem 4.4 *Let $A \in \text{Mat}^{n \times N}$, $B \in \text{Mat}^{N \times N}$, $D \in \text{Mat}^{n \times n}$, $\mathbf{f} \in \mathbf{C}^n$, $\mathbf{g} \in \mathbf{C}^N$ be such that the rank-1 condition (4.28) holds and $F \in \text{Mat}^{l \times n}$, $C \in \text{Mat}^{l \times N}$ satisfy the non-singularity condition*

$$\det(FAC^T) \neq 0. \quad (4.32)$$

The τ -function $\tau_{(A,B,C,D,F)}^f(\mathbf{t})$ is

$$\tau_{(A,B,C,D,F)}^f(\mathbf{t}) = \det(FAC^T) \tau_{W(\mathbf{f}, \mathbf{g}, B, D, M)}(\mathbf{t}), \quad (4.33)$$

where $W(\mathbf{f}, \mathbf{g}, B, D, M)$ is the subspace with geometric affine coordinates, associated to the quintuple $(\mathbf{f}, \mathbf{g}, B, D, M)$, with M given by

$$M = C^T(FAC^T)^{-1}F. \quad (4.34)$$

Proof. As a consequence of the decoupling integral formula (4.31), the first finite determinant expressing $\tau_{W(\mathbf{f}, \mathbf{g}, B, D, M)}(\mathbf{t})$ in (4.14) simplifies to the form

$$\begin{aligned}\tau_{W(\mathbf{f}, \mathbf{g}, B, D, C^T(FAC^T)^{-1}F)}(\mathbf{t}) &= \det \left(\mathbf{I}_n + e^{-\sum t_i (D^T)^i} \left(Ae^{\sum t_i B^i} - e^{\sum t_i (D^T)^i} A \right) C^T (FAC^T)^{-1} F \right)_{n \times n} \\ &= \det \left(\mathbf{I}_l + Fe^{-\sum t_i (D^T)^i} \left(Ae^{\sum t_i B^i} - e^{\sum t_i (D^T)^i} A \right) C^T (FAC^T)^{-1} \right)_{l \times l} \\ &= \det(FAC^T)^{-1} \det \left(Fe^{-\sum t_i (D^T)^i} Ae^{\sum t_i B^i} C^T \right),\end{aligned}\quad (4.35)$$

where the Weinstein–Aronszajn identity was used in the first equality.

Q.E.D.

Remark 4.1 As an alternative representation of the τ -function $\tau_{(A, B, C, D, F)}^f(\mathbf{t})$, we can choose the subspace $W(\mathbf{g}, \mathbf{f}, D^T, B^T, -F^T(CA^T F^T)^{-1}C)$, for which

$$\tau_{(A, B, C, D, F)}^f(\mathbf{t}) = \det(FAC^T) \tau_{W(\mathbf{g}, \mathbf{f}, D^T, B^T, -F^T(CA^T F^T)^{-1}C)}(-\mathbf{t}). \quad (4.36)$$

From Corollary 4.3 follows

Corollary 4.5 The following Schur function expansion holds:

$$\tau_{(A, B, C, D, F)}^f(\mathbf{t}) = \sum_{(\mathbf{a}|\mathbf{b})} \det(\mathbf{g}^T(-B)^{b_j} C^T (FAC^T)^{-1} F (D^T)^{a_i} \mathbf{f}) S_{(\mathbf{a}|\mathbf{b})}(\mathbf{t}). \quad (4.37)$$

By Theorem 4.1 we have the inclusions

Corollary 4.6

$$r_D(z) \mathcal{H}_+ \subset \mu_D(z) \mathcal{H}_+ \subset W(\mathbf{f}, \mathbf{g}, B, D, M) \subset \frac{1}{\mu_B(z)} \mathcal{H}_+ \subset \frac{1}{r_B(z)} \mathcal{H}_+, \quad (4.38)$$

$$r_B(z) \mathcal{H}_+ \subset \mu_B(z) \mathcal{H}_+ \subset W(\mathbf{g}, \mathbf{f}, D^T, B^T, -E^T(CAF^T)^{-1}C) \subset \frac{1}{\mu_D(z)} \mathcal{H}_+ \subset \frac{1}{r_D(z)} \mathcal{H}_+. \quad (4.39)$$

Specializing to the case when $l = n$ and $F = \mathbf{I}_n$, we obtain:

Corollary 4.7 The Gekhtman–Kasman τ -function

$$\tau_{(A, B, C)}^f(\mathbf{t}) = \det \left(Ae^{\sum t_i B^i} C^T \right) \quad (4.40)$$

with nonvanishing constant term

$$\det(AC^T) \neq 0 \quad (4.41)$$

can be written in terms of the τ -functions of either of the subspaces $W(\mathbf{f}, \mathbf{g}, B, D, C^T(AC^T)^{-1})$ or $W(\mathbf{g}, \mathbf{f}, D^T, B^T, -(CA^T)^{-1}C)$ as

$$\tau_{(A,B,C)}^f(\mathbf{t}) = \begin{cases} e^{\sum t_i \text{Tr}(D^i)} \det(AC^T) \tau_{W(\mathbf{f}, \mathbf{g}, B, D, C^T(AC^T)^{-1})}(\mathbf{t}), \\ e^{\sum t_i \text{Tr}(D^i)} \det(AC^T) \tau_{W(\mathbf{g}, \mathbf{f}, D^T, B^T, -(CA^T)^{-1}C)}(-\mathbf{t}). \end{cases} \quad (4.42)$$

The presence of the gauge factor $e^{\sum t_i \text{Tr}(D^i)}$ in (4.42), when compared with the linear exponential factor appearing in (1.7), implies that the effective subspace $\tilde{W}(\mathbf{f}, \mathbf{g}, B, D, C^T(AC^T)^{-1})$ that corresponds to $\tau_{(A,B,C)}^f(\mathbf{t})$ is given by

$$\tilde{W}(\mathbf{f}, \mathbf{g}, B, D, C^T(AC^T)^{-1}) := \frac{z^n}{r_D(z)} W(\mathbf{f}, \mathbf{g}, B, D, C^T(AC^T)^{-1}), \quad (4.43)$$

which satisfies the inclusion chain

$$z^n \mathcal{H}_+ \subset \tilde{W}(\mathbf{f}, \mathbf{g}, B, D, C^T(AC^T)^{-1}) \subset \frac{z^n}{r_B(z)r_D(z)} \mathcal{H}_+. \quad (4.44)$$

The subspace $\tilde{W}(\mathbf{f}, \mathbf{g}, B, D, C^T(AC^T)^{-1})$ thus belongs to the sub-Grassmannian Gr_1 of ref. ([16]).

5 Concluding remarks

We have shown how KP τ -functions of the finite determinantal form (1.10) and (1.15) may be derived through a subquotienting procedure applied to KP flows on the infinite dimensional Grassmannians of Sato, Segal and Wilson [14, 15, 16], viewed as projections to linear exponential flows on finite Grassmannians. These solutions were also expressed as fermionic operator vacuum expectation values of suitably defined products of creation and annihilation operators. This approach described here may be applied more generally to other integrable hierarchies, such as discrete KP, MKP chains of τ -functions, depending both on continuous and discrete flow variables, and the 2-Toda lattice hierarchy, resulting in analogous finite determinantal τ -functions.

The cases treated here are all “generic” solutions to the rank-1 condition (1.13), in the sense that the eigenvalues of the pair of matrices B and D are required to be distinct, and

the generic conditions (3.82) satisfied. In the sequel [2], these conditions will be dropped, and a complete classification of all solutions of the rank-1 condition (1.13), for all possible pairs of vectors \mathbf{f}, \mathbf{g} and matrices B, D determined, up to natural equivalence within orbits of the stabilizers of B and D .

A Appendix: Proof that $\tau_{(A,B,C,D,F)}^f(\mathbf{t})$ satisfies the Hirota bilinear equations

A concise way to express the Hirota bilinear equations for a KP τ -function $\tau(\mathbf{t})$ is to define an associated family of 2-forms $\xi(z_1, z_2, z_3, z_4, \mathbf{t})$ in \mathbf{C}^4 , depending on four complex parameters (z_1, z_2, z_3, z_4) together with the KP flow parameters $\mathbf{t} = (t_1, t_2, \dots)$:

$$\xi(z_1, z_2, z_3, z_4, \mathbf{t}) = \sum_{i,j=1}^4 \xi_{ij} e_i \wedge e_j \in \Lambda^2 \mathbf{C}^4, \quad (\text{A.1})$$

where

$$\xi_{ij} := (z_i - z_j) \tau(\mathbf{t} - [z_i^{-1}] - [z_j^{-1}]). \quad (\text{A.2})$$

The Hirota bilinear equations are then equivalent [14, 15] to the single Plücker relation defining the image of the Grassmannian $\text{Gr}_2(\mathbf{C}^4)$ in $\mathbf{P}(\Lambda^2 \mathbf{C}^4)$ under the Plücker map:

$$\begin{aligned} \mathfrak{Pl} : \text{Gr}_2(\mathbf{C}^4) &\rightarrow \mathbf{P}(\Lambda^2 \mathbf{C}^4) \\ \mathfrak{Pl} : \text{span}(W_1, W_2) &\mapsto [W_1 \wedge W_2], \quad W_1, W_2 \in \mathbf{C}^4. \end{aligned} \quad (\text{A.3})$$

i.e. the decomposability condition

$$\xi \wedge \xi = 0, \quad (\text{A.4})$$

$$\xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23} = 0, \quad (\text{A.5})$$

satisfied identically in the parameters (z_1, z_2, z_3, z_4) for all \mathbf{t} . We will prove this holds for the τ -function $\tau_{(A,B,C,D,F)}^f(\mathbf{t})$ by explicitly computing the vectors $W_1, W_2 \in \mathbf{C}^4$.

Define $\xi^{(A,B,C,D,F)}(z_1, z_2, z_3, z_4, \mathbf{t})$ as in (A.2) with $\tau = \tau_{(A,B,C,D,F)}^f$

$$\xi_{ij}^{(A,B,C,D,F)}(z_1, z_2, z_3, z_4, \mathbf{t}) := (z_i - z_j) \tau_{(A,B,C,D,F)}^f(\mathbf{t} - [z_i^{-1}] - [z_j^{-1}]), \quad (\text{A.6})$$

and also define

$$F(\mathbf{t}) := F e^{-\sum_{i=1}^{\infty} t_i (D^T)^i}, \quad C^T(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i B^i} C^T,$$

$$M^T(\mathbf{t}) := C^T(\mathbf{t}) (F(\mathbf{t})AC^T(\mathbf{t}))^{-1} F(\mathbf{t}), \quad (\text{A.7})$$

which is well defined provided $\tau_{(A,B,C,D,F)}^f(\mathbf{t}) \neq 0$. The following lemma shows that eq. (A.5) is satisfied by $\tau_{(A,B,C,D,F)}^f(\mathbf{t})$.

Lemma A.1

$$\xi^{(A,B,C,D,F)} = W_1 \wedge W_2 \quad (\text{A.8})$$

where

$$\begin{aligned} W_1 &:= \tau_{(A,B,C,D,F)}^f(H(z_1), H(z_2), H(z_3), H(z_4)) \\ W_2 &:= (G(z_1), G(z_2), G(z_3), G(z_4)) \end{aligned} \quad (\text{A.9})$$

with the rational functions $H(z)$, $G(z)$ defined by

$$\begin{aligned} H(z) &:= \tau_{(A,B,C,D,F)}^f(\mathbf{t}) (1 + \mathbf{g}^t M(\mathbf{t})(D^T - z \mathbf{I}_n)^{-1} \mathbf{f}) \\ G(z) &:= z + \mathbf{g}^t B M(\mathbf{t})(D^T - z \mathbf{I}_n)^{-1} \mathbf{f}. \end{aligned} \quad (\text{A.10})$$

Proof. Since

$$e^{-\sum_{i=1}^{\infty} \frac{1}{i} (B/z)^i} = \mathbf{I}_N - B/z, \quad e^{\sum_{i=1}^{\infty} \frac{1}{i} (D^T/z)^i} = (\mathbf{I}_n - D^T/z)^{-1}, \quad (\text{A.11})$$

we have

$$\begin{aligned} \xi_{ij}^{(A,B,C,D,F)} &= (z_i - z_j) \det(F(\mathbf{t})(D^T - z_i \mathbf{I}_n)^{-1}(D^T - z_j \mathbf{I}_n)^{-1} A(B - z_i \mathbf{I}_N)(B - z_j \mathbf{I}_N) C^T(\mathbf{t})) \\ &= (z_i - z_j) \det(F(\mathbf{t})AC^T(\mathbf{t})) \\ &\quad + (D^T - z_i \mathbf{I}_n)^{-1}(D^T - z_j \mathbf{I}_n)^{-1}(D^T - z_i - z_j) \mathbf{f} \mathbf{g}^T C^T(\mathbf{t}) \\ &\quad + (D^T - z_i \mathbf{I}_n)^{-1}(D^T - z_j \mathbf{I}_n)^{-1} \mathbf{f} \mathbf{g}^T B C^T(\mathbf{t}) \\ &= (z_i - z_j) \det(F(\mathbf{t})AC^T(\mathbf{t})) \det(\mathbf{I}_l) \\ &\quad + (D^T - z_i \mathbf{I}_n)^{-1}(D^T - z_j \mathbf{I}_n)^{-1}(D^T - z_i - z_j) \mathbf{f} \mathbf{g}^T M^T(\mathbf{t}) \\ &\quad + (D^T - z_i \mathbf{I}_n)^{-1}(D^T - z_j \mathbf{I}_n)^{-1} \mathbf{f} \mathbf{g}^T B M^T(\mathbf{t}) \\ &= (z_1 - z_2) \tau_{(A,B,C,D,F)}^f(\mathbf{t}) \det(\mathbf{I}_l + \mathbf{f}_1^T(z_1, z_2) \mathbf{g}_1^T + \mathbf{f}_2^T(z_1, z_2) \mathbf{g}_2^T) \end{aligned} \quad (\text{A.12})$$

where

$$\begin{aligned} \mathbf{f}_1 &:= (D^T - z_i \mathbf{I}_n)^{-1}(D^T - z_j \mathbf{I}_n)^{-1}(D^T - z_i - z_j) \mathbf{f}, & \mathbf{g}_1^T &:= \mathbf{g}^T M^T(\mathbf{t}), \\ \mathbf{f}_2 &:= (D^T - z_i \mathbf{I}_n)^{-1}(D^T - z_j \mathbf{I}_n)^{-1} \mathbf{f}, & \mathbf{g}_2^T &:= \mathbf{g}^T B M^T(\mathbf{t}). \end{aligned} \quad (\text{A.13})$$

In the second line of (A.12) we have used the rank-1 condition (1.13) in the form

$$A(B - z \mathbf{I}_N) = (D - z \mathbf{I}_n)A + \mathbf{f} \mathbf{g}^T. \quad (\text{A.14})$$

Using the Weinstein-Aronszajn identity, we can rewrite this as a 2×2 determinant

$$\begin{aligned}\xi_{ij}^{(A,B,C,D,F)} &= (z_1 - z_2) \tau_{(A,B,C,D,F)}^f(\mathbf{t}) \det \begin{pmatrix} 1 + \mathbf{g}_1^T \mathbf{f}_1 & \mathbf{g}_1^T \mathbf{f}_2 \\ \mathbf{g}_2^T \mathbf{f}_1 & 1 + \mathbf{g}_2^T \mathbf{f}_2 \end{pmatrix} \\ &= \tau_{(A,B,C,D,F)}^f(\mathbf{t}) \det \begin{pmatrix} H(z_i) & H(z_j) \\ G(z_i) & G(z_j) \end{pmatrix},\end{aligned}\tag{A.15}$$

where the last line follows from elementary column operations. This is equivalent to Eq. (A.8). **Q.E.D.**

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